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Abstract

This paper suggests incorporating investor probability weighting and the default risk of individual firms into a consumption-based asset pricing model. The extended model provides a unified solution for several anomalous patterns observed on financial markets. The analysis addresses not only widely-recognized asset pricing puzzles, such as the equity premium puzzle, but also less studied anomalies on financially distressed stocks. The simulation, under which the model is calibrated according to U.S. historical data, shows the combination of mild overweighting of probability on tail events and nonlinearity of equity values caused by default risk has the potential to resolve these patterns.

Keywords: asset pricing puzzles; consumption-based asset pricing; distressed stock; probability weighting function; default risk; business time

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1 Introduction

Since the seminal study by Lucas (1978), the consumption-based asset pricing approach has offered a theoretical deep understanding on the mechanism of asset pricing and investor behavior, and has been used for numerous empirical analyses. On the other hand, this approach produces a number of counterfactual predictions, known as *asset pricing puzzles*. Although many researchers have attempted to solve the puzzles, they are still controversial. As such, the aim of this paper is to provide one possible explanation for the puzzles by proposing a version of the consumption-based model that includes investor probability weighting and the default risks of individual firms. To the best of my knowledge, this paper is the first to take both these factors into account in a consumption-based model.

As a result, in the theoretical section of this paper, I extend the standard consumption-based asset pricing model from the following three dimensions. First, I consider investor subjective probability weighting. In the model, the representative investor is assumed not to use objective probability when evaluating assets but rather a transformed probability obtained from the objective probability via a probability weighting function. The transformed probability is thus called *subjective probability*. The probability weighting function is recognized as an aspect of the cumulative prospective theory of Tversky and Kahneman (1992), which is a modified version of the prospective theory of Kahneman and Tversky (1979). The S-shaped probability weighting underestimates the tail events of the outcomes distribution, whereas the inverse S-shaped weighting overweighs them. A number of experimental studies, such as Tversky and Kahneman (1992), Wu and Gonzalez (1996), Prelec (1998), and Berns et al. (2007), supports the latter shape. Consequently, the probability weighting function could thus help understand investor behavior on the financial markets. Second, I incorporate Merton's (1974) structural credit risk modeling in the consumption-based model. Structural modeling employs the contingent claim approach to evaluate firm debt and equity, which are regarded as put and call options written on firm value, respectively. It also explicitly expresses the relationship between the default risk and capital structure of a firm. Thereby, the financial leverage effect on firm debt and equity is naturally described in the model. In fact, the incorporation of structural credit risk modeling into the consumption-based model has been less studied in macro financial literature, despite numerous studies on the structural modeling of credit market analysis. Finally, the proposed model allows for non-normally distributed random shocks in the economy. In this respect, I consider a time-varying information arrival rate called *business time* that subordinates a Brownian motion. As a result, the Brownian motion is transformed into a pure jump process with highly frequent and small jumps. The dynamic nature of the stochastic process is thus capable of capturing fat tails for the consumption growth rate distributions. This modeling is parsimonious, but tractable with only one additional parameter. It allows for simple and robust calibration to sample data. The concept of business time is closely related to subjective probability weighting. As such, investor perspective of consumption against the variation in business time endogenously determines whether he/she is cautious about rare events.

The first half of the simulation explores five major puzzles as follows. (i) *Equity premium puzzle*: Fluctuations observed in aggregate consumption predict a too small equity premium (Mehra & Prescott, 1985). (ii) *Excess volatility puzzle*: Stock volatility far exceeds dividend volatility (Shiller, 1981). (iii) *Stock return predictability*: Future aggregate stock returns are predictable using the current level of price-dividend ratios (Campbell & Shiller, 1988). (iv) *Risk-free rate puzzle*: Observed interest rates seem much smaller than warranted by a model with reasonable parameters (Weil, 1989). (v) *Credit spread puzzle*: Credit spreads of corporate bonds are seemingly too high relative to historical default probabilities (Almeida & Philippon, 2007). A vast literature has already explored a modified version of the consumption-based model to explain these puzzles, and the proposed model attempts to tackle this challenge as

well.

The second half of the simulation sheds light on the anomalous patterns observed for financially distressed stocks, which present a complex picture. Dichev (1998), Griffin and Lemmon (2002), Campbell et al. (2008), and others report that firms with higher default probability tend to have lower future stock returns. Moreover, distressed stocks occasionally earn negative average excess returns. However, this observation contradicts the fundamental principle of investment theory that investors demand higher premiums from riskier assets. Campbell et al. (2008) and Avramov et al. (2009) find that distressed stocks have large CAPM betas, despite being weakly correlated with a market portfolio, while their Jensen's alphas are negative. Further, Campbell et al. (2008) also document that extremely high volatility for distressed stocks cannot be diversified by constructing portfolios. This indicates the presence of covariation among the returns on distressed stocks. These anomalously empirical patterns posit a further challenge for this study.

The analysis lead to the following findings, some of which provide plausible interpretations, in line with the standard theory of finance, and others novel interpretation. As a prerequisite for the findings, the representative investor must have *mild* overweighting of his/her subjective probability on tail events, which is consistent with past experimental research in behavioral economics. Another key point is the default risks of individual firms. First, investor cautiousness about tail events lowers the risk-free interest rate because he/she demands significant precautionary savings. Given such an investor, increasing risk aversion leads to a low interest rate. Second, financial leverage raises equity risk premiums and volatility for individual firms. Consequently, the market portfolio in the model earns higher expected excess return than warranted by the standard model. Therefore, the financial leverage effect on firms' equity reconciles the equity premium and excess volatility puzzles. Third, the combination of probability weighting and default risk flawlessly solves the anomalous cross-sectional situation that expected returns on distressed stocks are decreasing with the deterioration of default risk. As shown in the analysis, this novel finding is closely related to the shape of the pricing kernel and optionality of distressed stocks. The investor is willing to hold a distressed stock despite a low expected return, because he/she think of the stock as a *lottery*. Finally, the anomalous correlation structure among distressed stocks is reproduced by the strict nonlinearity of returns on the distressed stocks against the market portfolio. The nonlinearity causes extremely high CAPM betas and negative Jensen's alpha for the distressed stocks as well. In the model, the seemingly anomalous patterns mentioned above are no longer puzzles but emerge naturally. Unfortunately, the stock return predictability and credit spread puzzle are not yet settled by the analysis.

Unlike the present work, past studies address neither major asset pricing puzzles, such as the equity premium puzzle, nor the anomalous patterns observed for financially distressed stocks. There has been a wide range of studies that propose alternative asset pricing models to explain major puzzles. A list of the prominent examples of these alternative models is as follows: recursive utility (Epstein & Zin, 1989; Weil, 1989), habit formation (Campbell & Cochrane, 1999), long-run risk (Bansal & Yaron, 2004; Bansal et al., 2012), rare disasters (Reitz, 1988; Barro, 2006), heterogeneous preferences (Gârleanu & Panageas, 2015), probability mistakes (Shiller, 2014), idiosyncratic risk (Constantinides & Duffie, 1996), and ambiguity aversion (Hansen & Sargent, 2001). Within the class of consumption-based asset pricing frameworks, the proposed model may be another workable framework, along with the existing consumption-based models listed above. Cochrane (2017) presents a survey of various types of asset pricing models in macro-finance and their relationship to asset pricing puzzles.

By contrast, relatively few studies address the construction of a model to explain anomalous patterns in distressed stocks. For example, Garlappi and Yan (2011) explicitly introduce finan-

cial leverage into an equity valuation model. They demonstrate the possibility of shareholder recovery upon financial distress causes a seemingly anomalous relation between an expected stock return and default probability. However, they follow a partial equilibrium approach. Relevant literature is discussed in more detail in Section 3.5.

The remainder of this paper proceeds as follows. Section 2 presents the model and develops the main intuitions for the effect of investor probability weighting and default risk of individual firms on asset pricing. Section 3 discusses calibration and simulation. Section 4 concludes the paper. Further, the Appendices contain all proofs and some technical supplements.

2 Model

This section aims to construct a version of the consumption-based asset pricing model to explore solutions for asset pricing puzzles. The model developed below allows for investor subjective probability weighting and the default risk of firms simultaneously. For simplicity, the model is assumed to be driven by only one common factor, without any idiosyncratic factors.

2.1 Basic Setup

Assume an endowment economy satisfying a no-arbitrage condition. Let C_t be the *objective* aggregate consumption process. It is assumed that its log growth rate R_t follows stochastic process

$$R_t := \log \frac{C_t}{C_0} = \mu t + \sigma W(\tau_t), \quad (1)$$

where μ and σ are constants and $W(t)$ is a standard Brownian motion. Here, τ_t is the *business time* at time t , contrasting with calendar time t , which is also known as the *subordinator* in stochastic calculus. That is, τ_t is an increasing Lévy process,¹ independent of $W(t)$, such that $\mathbb{E}^\mathbb{P}[\tau_t] = t$ for all $t \geq 0$, where $\mathbb{E}^\mathbb{P}[\cdot]$ denotes an unconditional expectation operator under the *objective probability measure* \mathbb{P} . Roughly, this time scale can be thought of as the integrated rate of information arrival. Another interpretation of the business time is the integrated stochastic volatility of $W(t)$. The economy is rapidly changing when $\tau_t > t$, and vice versa. Panels A and B in Figure 1 visualize sample paths of the business time. For a more detailed interpretation of business time, see Cont and Tankov (2004) and Carr et al. (2003). The mean, variance, and skewness of the log growth rate R_t are μt , $\sigma^2 t$, and zero, respectively, while the excess kurtosis equals $3\text{Var}[\tau_1]/t$. Therefore, the stochastic business time makes the log consumption growth rate non-Gaussian distributed. Hereafter, $\mathcal{L}(\theta)$ denotes the Laplace exponent of business time τ_t , defined as

$$\mathcal{L}(\theta) := \frac{1}{t} \log \mathbb{E}^\mathbb{P} [e^{\theta \tau_t}],$$

with parameter $\theta \in \mathcal{D}_\tau$, where domain \mathcal{D}_τ is a subset of the complex plane, such that the Laplace exponent is well defined. Note that the Laplace exponent is a deterministic function of θ , independent of time t . See Appendix A.2 on the detailed properties of the Laplace exponent.

There is an infinitely lived representative investor in the economy, who has a power utility function over aggregate consumption. However, his/her subjective view about the future growth rates of aggregate consumption does not follow the stochastic process (1), but

$$R_t^S := \log \frac{C_t^S}{C_0^S} = R_t + \omega (\tau_t - t) = \mu t + \sigma W(\tau_t) + \omega (\tau_t - t),$$

¹See Appendix A for the definition and fundamental properties of Lévy processes.

where C_t^S denotes the investor's *subjective* consumption process and ω is called the subjective parameter. For $\omega < 0$, the investor believes consumer sentiment will deteriorate due to rapid changes in the economy, and vice versa. Only when $\omega = 0$, subjective future consumption coincides with the objective one at any time. The subjective mean of a consumption growth rate is confined to μt in any case, although the other subjective moments, such as variance, skewness, and kurtosis, might vary from the objective ones. Consequently, the investor's expected utility takes the form

$$\mathbb{E}^{\mathbb{P}} \left[\int_0^\infty e^{-\delta t} \frac{(C_t^S)^{1-\gamma}}{1-\gamma} dt \right],$$

with the rate of time preference δ and the relative risk aversion γ being non-negative constants.

Following Lemma A.3 in Appendix A.2, both log consumption growth rate processes R_t and R_t^S turn out to be Lévy processes, also called *subordinated Brownian motions*. In many textbooks on continuous-time asset pricing theory, the stochastic process for describing market uncertainty is typically assumed to be a Brownian motion, which becomes a special case when $\tau_t = t$ for all $t \geq 0$ in this framework, belonging to the class of Lévy processes. Although a Brownian motion might perform as a benchmark model in this paper, there are numerous advantages to adopting such Lévy processes for more realistic modeling. Lévy processes provide a flexible framework for describing discontinuous fluctuations in aggregate consumption. For example, the variance gamma process proposed by Madan and Seneta (1990), which has a gamma process as the business time, is a finite variation process with infinite but relatively low activity of small jumps. On the other hand, the normal inverse Gaussian process originally developed by Barndorff-Nielsen (1997), which equips an inverse Gaussian process as the business time, is an infinite variation process with high activity of small jumps. Since such processes have non-Gaussian distributed increments, they can produce skewness and excess kurtosis of the log consumption distribution. In the theory of financial derivatives, a large number of studies used Lévy processes categorized within subordinated Brownian motions to model the dynamics of underlying asset prices. For instance, see Cont and Tankov (2004), Schoutens (2003), Rachev et al. (2011). On the other hand, regarding macro-finance literature, few studies explicitly address the stochastic business time. For instance, Yamazaki (2018) examines investor sensitivity of business time on stock index option prices. For the simulation, the variance gamma and normal inverse Gaussian processes will be considered.

2.2 Stochastic Discount Factors and Probability Weighting

Investor's subjective stochastic discount factor \mathcal{M}_t^S can be decomposed into two parts as follows:

$$\mathcal{M}_t^S := e^{-\delta t} \left(\frac{C_t^S}{C_0^S} \right)^{-\gamma} = \mathcal{M}_t^O \frac{d\mathbb{P}^S}{d\mathbb{P}} \Big|_t, \quad (2)$$

where \mathcal{M}_t^O denotes the objective stochastic discount factor defined by

$$\mathcal{M}_t^O := e^{-\bar{\delta} t} \left(\frac{C_t}{C_0} \right)^{-\gamma}, \quad (3)$$

with the adjusted time preference parameter $\bar{\delta} := \delta - \gamma\omega - \mathcal{L}(-\gamma\omega)$ and

$$\frac{d\mathbb{P}^S}{d\mathbb{P}} \Big|_t := \frac{e^{-\gamma\omega(\tau_t-t)}}{\mathbb{E}^{\mathbb{P}} [e^{-\gamma\omega(\tau_t-t)}]} = \exp \{ -\gamma\omega\tau_t - \mathcal{L}(-\gamma\omega)t \}, \quad (4)$$

which is the Radon-Nikodym derivative that defines investor's *subjective probability* \mathbb{P}^S .

Next, *risk-neutral probability* \mathbb{Q} is defined by the Radon-Nikodym derivative as

$$\left. \frac{d\mathbb{Q}}{d\mathbb{P}} \right|_t := \frac{\mathcal{M}_t^S}{\mathbb{E}^{\mathbb{P}}[\mathcal{M}_t^S]} = e^{rt} \mathcal{M}_t^S = e^{rt} \mathcal{M}_t^{\mathcal{O}} \left. \frac{d\mathbb{P}^S}{d\mathbb{P}} \right|_t = \left. \frac{d\mathbb{Q}}{d\mathbb{P}^S} \right|_t \times \left. \frac{d\mathbb{P}^S}{d\mathbb{P}} \right|_t,$$

where r denotes the risk-free rate in the economy, whose formal expression will be derived in Section 2.4, and

$$\left. \frac{d\mathbb{Q}}{d\mathbb{P}^S} \right|_t := \frac{\mathcal{M}_t^{\mathcal{O}}}{\mathbb{E}^S[\mathcal{M}_t^{\mathcal{O}}]} = e^{rt} \mathcal{M}_t^{\mathcal{O}},$$

where $\mathbb{E}^S[\cdot]$ denotes an unconditional expectation operator under subjective measure \mathbb{P}^S . As a result, the present value of an arbitrary cash flow paid at time t , which is assumed to be a random variable X_t , has three equivalent representations:

$$\mathbb{E}^{\mathbb{P}}[\mathcal{M}_t^S X_t] = \mathbb{E}^S[\mathcal{M}_t^{\mathcal{O}} X_t] = \mathbb{E}^{\mathbb{Q}}[e^{-rt} X_t], \quad (5)$$

where $\mathbb{E}^{\mathbb{Q}}[\cdot]$ denotes the risk-neutral expectation operator. According to (2), this present value is also represented as

$$\int \mathcal{M}_t^{\mathcal{O}} X_t \left(\left. \frac{d\mathbb{P}^S}{d\mathbb{P}} \right|_t \right) d\mathbb{P}.$$

Therefore, the Radon-Nikodym derivative (4) can be thought of as the *probability weighting function* of the representative investor. It is worth noting subjective parameter ω determines not only the subjective view on future consumption, but also investor probability weighting when evaluating any asset.

Considering probability theory, each change of probability measure in (5) can be viewed as the Esscher transform, as shown below. The Esscher transform alters the drift term and jump structure of a Lévy process, whereas the variance component is unchanged. Appendix A.3 briefly discusses the Esscher transform by an arbitrary Lévy process and provides general representations for the transformed drift term and jump structure.

- The change of measure from objective probability \mathbb{P} to subjective probability \mathbb{P}^S via the probability weighting function (4) is the Esscher transform by business time τ_t with parameter $-\gamma\omega$.
- The change of measure from subjective probability \mathbb{P}^S to risk-neutral probability \mathbb{Q} via the objective stochastic discount factor (3) is the Esscher transform by objective consumption growth rate R_t with parameter $-\gamma$.
- The change of measure from objective probability \mathbb{P} to risk-neutral probability \mathbb{Q} via the subjective stochastic discount factor (2), which is the product of the above two Esscher transforms, is the Esscher transform by subjective consumption growth rate R_t^S with parameter $-\gamma$.

2.3 Consumption Growth Rate Distributions

The characteristic functions of consumption growth rate distributions under the three probability measures \mathbb{P} , \mathbb{P}^S , and \mathbb{Q} are useful as follows. First, almost all computation formulas in this paper are represented as the generalized Fourier transform of them (See Appendix C for

the derived formulas). Second, the characteristic functions are tractable to compute the fundamental statistics of R_t , such as standard deviation, skewness, and excess kurtosis. Third, the cumulative distribution and density functions of R_t can be immediately obtained from so-called Lévy's inversion theorem. Recall the objective consumption growth rate process R_t is a Lévy process. Additionally, a transformed Lévy process by the Esscher transform also belongs to the class of Lévy processes. Therefore, applying the Lévy-Khintchine formula stated in Lemma A.1, the characteristic function of a distribution of R_t can be expressed as

$$\Phi_{R_t}^*(\theta) := \mathbb{E}^* [e^{i\theta R_t}] = e^{t\varphi_R^*(\theta)}, \quad (6)$$

where superscript $*$ takes an arbitrary probability measure of \mathbb{P} , \mathbb{P}^S , or \mathbb{Q} . From Lemma A.3, the characteristic exponent of R_t under the objective probability \mathbb{P} takes the form

$$\varphi_R^{\mathbb{P}}(\theta) = i\theta\mu + \mathcal{L}(-\theta^2\sigma^2/2). \quad (7)$$

The subsequent lemma provides the general representations of the characteristic exponents of R_t under subjective and risk-neutral probability measures.

Lemma 1 *The characteristic exponent of R_t under subjective probability \mathbb{P}^S is given by*

$$\varphi_R^S(\theta) = i\theta\mu + \mathcal{L}(-\theta^2\sigma^2/2 - \gamma\omega) - \mathcal{L}(-\gamma\omega). \quad (8)$$

Moreover, the characteristic exponent of R_t under risk-neutral probability \mathbb{Q} is given by

$$\varphi_R^{\mathbb{Q}}(\theta) = i\theta\mu + \mathcal{L}((i\theta - \gamma)^2\sigma^2/2 - \gamma\omega) - \mathcal{L}(\gamma^2\sigma^2/2 - \gamma\omega). \quad (9)$$

Proof of Lemma 1: See Appendix B.1. \square

In the following, I also use the moment generating function of the distribution of R_t under arbitrary probability, which is denoted by $\Psi_{R_t}^*(\theta)$ with parameter $\theta \in \mathcal{D}_R$, where domain \mathcal{D}_R is a subset of the real line, such that the function is well defined.

Note the subjective distribution of R_t determined by characteristic exponent (8) differs from the objective distribution of R_t^S . The former is linked to relative risk aversion γ , but the latter is not, and its characteristic exponent is given by

$$\varphi_{R^S}^{\mathbb{P}}(\theta) = i\theta(\mu - \omega) + \mathcal{L}(-\theta^2\sigma^2/2 + i\omega\theta).$$

Subjective parameter ω causes the objective distribution of the subjective consumption growth rate R_t^S to be skewed. The sign of ω corresponds to sign of the skewness. Table A.2 exhibits the Laplace exponents and cumulants for the consumption growth rate process modeled by a Brownian motion, variance gamma process, and normal inverse Gaussian process. The forms of these processes are invariant to the Esscher transforms, but their parameters are transformed. Table A.3 shows the transformed parameters corresponding to each probability measure. Interestingly, transformed parameter ω_S in Panel B of Table A.3 equals zero, and then the third cumulant in Table A.2 is also reduced to zero. This means the subjective distribution of the objective consumption growth rate is not skewed. The subjective parameter affects only the standard deviation and kurtosis in the subjective distribution.

Several extant studies investigate the relationship between objective and risk-neutral distributions. For example, Yamazaki (2018) examines the impact of stochastic business time on a risk-neutral distribution of stock index returns, but only for a risk-averse investor with non-biased probability weighting ($\gamma > 0$ and $\omega = 0$). His conclusion is that the tail of the risk-neutral density function is fatter than that of the objective density function, and the skewness

of the risk-neutral distribution is more negative than that of the objective distribution. This implies a risk-averse investor worries about downside risk of the stock index when evaluating any assets. Tables A.2 and A.3 indicate similar characteristics for consumption growth rates. This inclination becomes evident from the negative value of ω . An exceptional case is the Brownian motion. In this case, the mean of the risk-neutral distribution shifts from the mean of the objective distribution, whereas the shape of the distribution remains unchanged.

2.4 Interest Rate

The purpose of this subsection is to determine the risk-free interest rate at market equilibrium. I define the risk-free interest rate as $r(t) := -\frac{1}{t} \log B(t)$, where $B(t)$ denotes the present value of a zero-coupon risk-free bond that pays one unit at maturity t . Specifically, $r(t)$ is also called *yield to maturity*.

Proposition 1 (Interest Rate) *The risk-free interest rate is unvarying over maturity and given by*

$$r = \delta + \gamma\mu - \gamma\omega - \mathcal{L}(\gamma^2\sigma^2/2 - \gamma\omega). \quad (10)$$

Proof of Proposition 1: See Appendix B.2. □

As previously mentioned, the consumption growth rate R_t is a Lévy process even under risk-neutral probability. Due to the stationary increment property of Lévy processes, in this framework, the risk-free interest rate is constant over time and the yield curve is always flat.

Up to the second term on the right-hand side of (10), $\delta + \gamma\mu$ is called the *steady-state interest rate* in the field of the deterministic neoclassical growth modeling, while the remaining terms, $-\gamma\omega - \mathcal{L}(\gamma^2\sigma^2/2 - \gamma\omega)$, are known as the *precautionary savings term*. For example, if $\tau_t = t$ for all $t \geq 0$, which is a Brownian motion case, the precautionary savings term is written as $-\sigma^2\gamma^2/2$. In this case, the larger variance of the consumption growth rate decreases the precautionary savings term. Whereas higher risk aversion decreases the precautionary savings term, it increases the steady-state interest rate when expected growth rate μ is positive. This trade-off relation is controversial. In response to this incompatibility, Weil (1989) asserts the *risk-free rate puzzle* that the level of interest rates estimated by the standard asset pricing model is much higher than interest rates observed in the market. However, the variance of consumption growth rates estimated from sample data is too small to match the model-implied interest rate with historical average of interest rates.

The proposed model allows for a more flexible setting than asset pricing models based on Brownian motions. Proposition 1 indicates the model has two sources to reconcile the risk-free rate puzzle: the Laplace exponent of business time $\mathcal{L}(\theta)$ and subjective parameter ω . These are elements of the precautionary savings term, but not relevant to the steady-state interest rate. Barro (2006), Gabaix (2012), Wachter (2013), and others introduce *large negative jumps* into aggregate consumption as macroeconomic disasters, which cause the consumption growth rate distribution to be negatively skewed and fat-tailed. As a consequence of taking rare disasters into account, the precautionary savings term can take a sufficiently negative value, thus allowing for the reproduction of a plausible level of interest rate. On the other hand, this study introduces business time into a Brownian motion, which generates *frequent infinitesimal jumps* in aggregate consumption. Additionally, I am also concerned with how much subjective parameter ω affects the interest rate. These two components are expected to induce more precautionary savings. Therefore, Lemma A.2, which characterizes the Laplace exponent $\mathcal{L}(\theta)$, might be helpful to understand the behavior of the precautionary savings term.

2.5 Firm Value

In past studies on consumption-based asset pricing, the standard modeling of the cash flow generated by a firm linked with aggregate consumption is C_t^ν , where ν is a positive constant called the *leverage parameter*. For example, Abel (1999), Campbell (2003), Backus et al. (2011), and Wachter (2013) adopt such modeling. In this modeling, if increments of log consumption are independent and identically distributed, the cash flow growth rate is ν -times as large as the consumption growth rate at any time. This property may be convenient to the cash flow growth rate having a higher standard deviation than the consumption growth rate. However, there exists an inconvenience: The expected cash flow growth rate tends to be extremely high. To avoid this problem, firm cash flow Z_t is modeled as

$$\log \frac{Z_t}{Z_0} := \zeta t + \sigma_z W(\tau_t) = (\zeta - \mu\nu)t + \nu R_t, \quad (11)$$

where ζ is a constant called the *cash flow drift term*, and σ_z a positive constant such that $\sigma_z = \nu\sigma$. This modeling allows to calibrate cash flow drift term ζ and leverage parameter ν separately, both of which being firm-specific. Obviously, the log cash flow rate (11) is an affine transform of objective consumption growth rate (1). If $\zeta = \nu\mu$, which means the expected cash flow growth rate is ν -times as large as the expected consumption growth rate, model (11) coincides with standard modeling, that is, $Z_t = C_t^\nu$ at any time. According to cash flow modeling, firm value at time t can be represented as

$$V_t = \mathbb{E}_t^{\mathbb{P}} \left[\int_t^\infty \frac{\mathcal{M}_u^S}{\mathcal{M}_t^S} Z_u du \right] = \mathbb{E}_t^{\mathbb{Q}} \left[\int_t^\infty e^{-r(u-t)} Z_u du \right], \quad (12)$$

where $\mathbb{E}_t^*[\cdot]$ is a conditional expectation operator given information until time t with respect to an arbitrary probability measure.

Throughout the paper, the following assumption among model parameters is imposed upon all firms. As per the proof of Proposition 2, this assumption is the transversality condition on a firm value, that is, $0 < V_t < \infty$ at any time t .

Assumption 1 (Transversality Condition) *Model parameters satisfy the inequality*

$$\delta + \gamma\mu - \zeta - \gamma\omega - \mathcal{L}(\sigma^2(\nu - \gamma)^2/2 - \gamma\omega) > 0.$$

The next proposition presents a simple closed-form representation of the firm value given in (12) under Assumption 1.

Proposition 2 (Firm Value) *Firm value (12) is represented as*

$$V_t = \frac{Z_t}{\xi},$$

where

$$\xi := \delta + \gamma\mu - \zeta - \gamma\omega - \mathcal{L}(\sigma^2(\nu - \gamma)^2/2 - \gamma\omega).$$

Proof of Proposition 2: See Appendix B.3. □

Proposition 2 shows the log growth rate of a firm value coincides with the log growth rate of its cash flow. That is, the firm value drift term equals ζ and the standard deviation of the firm value growth rate equals σ_z . Consequently, firm-specific parameter σ_z is named *asset volatility*.

Included into the cash flow modeling (11), Proposition 2 indicates firm value is proportional to the ν -th power of the objective aggregate consumption. That is, it can be written as

$$V_t = \frac{Z_0}{\xi(C_0)^\nu} e^{(\zeta - \mu\nu)t} (C_t)^\nu. \quad (13)$$

The representation above is very important to interpreting debt and equity values, which will be subsequently discussed.

Note that the log growth rates of a firm value, its cash flow, and objective aggregate consumption are perfectly correlated with each other. Moreover, for any individual firms with different firm-specific parameters, the log growth rates of their cash flows and firm values have perfect positive correlations among them. Of course, this perfectly correlated structure is caused by all growth rates being modeled as simple linear regressions of common driving factor $W(\tau_t)$ under objective probability.

2.6 Debt Value and Credit Spread

I incorporate the structural credit risk modeling by Merton (1974) into a consumption-based asset pricing model for introducing default risk into the framework. The model takes default risk into account when evaluating a firm's debt and equity.

Assume a firm issues a single debt, carrying a promised terminal payoff F , which is interpreted as the face amount of a corporate zero-coupon bond maturing at time T . Postulate that a default event may only occur at the debt's maturity date, T . If the firm value at maturity V_T is less than or equal to the debt's face amount F , the firm is forced to default. Otherwise, the firm does not default. It is assumed that, when the default event occurs, a corporate law such as Chapter 11 in the United States or the Civil Rehabilitation Law of Japan invariably applies to the bankrupt firm, which still continues its business operations. Even after the default event, the firm generates ongoing cash flow. However, as a result of the default, the bankrupt firm must perform a 100% capital reduction, meaning the equity value becomes worthless and the firm pays only amount εV_T towards the debt, where $\varepsilon \in [0, 1]$ denotes a constant recovery rate equal to the bankruptcy costs for liquidating the firm. On a perfect market, $\varepsilon = 1$ holds for any firm. If the firm does not default until maturity, its debt is repaid in full at maturity.

Consequently, the payout of the firm's debt at maturity T can be written as $F\mathbf{1}_{\{V_T > F\}} + \varepsilon V_T\mathbf{1}_{\{V_T \leq F\}}$. Therefore, the present value of the debt $D(T)$ is expressed as

$$\begin{aligned} D(T) &= \mathbb{E}^\mathbb{P} \left[\mathcal{M}_T^S \left(F\mathbf{1}_{\{V_T > F\}} + \varepsilon V_T\mathbf{1}_{\{V_T \leq F\}} \right) \right] \\ &= e^{-rT} \mathbb{E}^\mathbb{Q} \left[F\mathbf{1}_{\{V_T > F\}} + \varepsilon V_T\mathbf{1}_{\{V_T \leq F\}} \right]. \end{aligned} \quad (14)$$

As a computation formula for the debt value (14), the following proposition is presented.

Proposition 3 (Debt Value) *Let $\Psi_{R_T}^\mathbb{Q}(\theta)$ be the risk-neutral moment generating function of R_T and*

$$A_{a,b}^\mathbb{Q}(k, T) := \mathbb{E}^\mathbb{Q} \left[e^{aR_T + bk} \mathbf{1}_{\{R_T > k\}} \right] \quad (15)$$

be the deterministic function, whose expression for computation is given by Lemma C.1 in Appendix C. Then, debt value (14) has the form

$$D(T) = e^{-\bar{r}T} V_0 \left[\varepsilon \Psi_{R_T}^\mathbb{Q}(\nu) - \varepsilon A_{\nu,0}^\mathbb{Q}(k, T) + A_{0,\nu}^\mathbb{Q}(k, T) \right],$$

with $\bar{r} := r + \mu\nu - \zeta$ and $k := \nu^{-1} \log(F/V_0) + (\mu - \zeta\nu^{-1})T$.

Since the seminal work of Merton (1974), a corporate bond has been identified with a default-free bond plus a short position in a put option written on firm value. Certainly, it is possible to interpret debt value (14) in this context. Additionally, recall that the firm value is proportional to the ν th power of the aggregate consumption according to (13). As a result, one can newly interpret a corporate bond as a default-free bond plus a short position in a *power put option* with ν -exponent written on aggregate consumption. In other words, the debt is a type of contingent claim on consumption expenditure. A power option is one of the financial derivatives whose payoff is based on the price of an underlying asset raised to a power. Such derivatives are designed to allow option holders to take a leveraged view on the underlying asset.

The credit spread of a corporate bond with maturity T denoted by $\text{spr}(T)$ takes the form

$$\begin{aligned}\text{spr}(T) &:= -\frac{1}{T} \log \frac{D(T)}{F} - r \\ &= \frac{1}{T} \left[\nu k - \log \left(\varepsilon \Psi_{R_T}^{\mathbb{Q}}(\nu) - \varepsilon A_{\nu,0}^{\mathbb{Q}}(k, T) + A_{0,\nu}^{\mathbb{Q}}(k, T) \right) \right].\end{aligned}\quad (16)$$

The objective probability of default until time T is given by

$$\mathbb{P}(V_T < F) = 1 - A_{0,0}^{\mathbb{P}}(k, T), \quad (17)$$

where $A_{a,b}^{\mathbb{P}}(k, T)$ is defined in the same manner as (15), but under objective probability \mathbb{P} . The levels of the credit spread and default probability depend on the *debt ratio*, defined as the ratio of the face amount of the debt to current firm value; F/V_0 ; the cash flow drift term ζ ; and leverage parameter ν , all of which are firm-specific. Besides these factors, the credit spread and default probability are also subject to the distribution of consumption growth rate R_T . Note that risk aversion parameter γ and subjective parameter ω , both of which determine investor characteristic, are only related to the credit spread.

The *credit spread puzzle*, named by Amato and Remolona (2003), can be regarded as incompatibility between the levels of credit spread and the default probability warranted by a structural credit model with reasonable parameters. In response to this puzzle, a number of authors have analyzed a wide range of structural firm value models. The model in this paper has two main different characteristics from existing models. One is introducing investor's subjective view on future economic conditions into the model, which is equivalent to probability weighting. The purpose of this study is to investigate how much subjective parameter ω affects credit spread when the probability of default is given. The other feature is the jump structure of stochastic modeling. Almost all past studies have incorporated rare and large jump risk premiums into their analyses (e.g., Collin-Dufresne et al., 2010; Driessen, 2005; and Cremers et al., 2008). By contrast, I consider highly frequent and small jumps in market fluctuations, occasionally without any continuous random shocks.

2.7 Equity Value

Assume a firm pays a dividend ηZ_t at each time t , where $\eta \in [0, 1]$ denotes the *dividend payout ratio* to cash flow. Since the liquidation value of equity at debt maturity T is $(V_T - F)^+$, the present value of equity $E(T)$ is represented as

$$\begin{aligned}E(T) &= \mathbb{E}^{\mathbb{P}} \left[\int_0^T \mathcal{M}_u^S \eta Z_u du + \mathcal{M}_T^S (V_T - F)^+ \right] \\ &= \mathbb{E}^{\mathbb{Q}} \left[\int_0^T e^{-ru} \eta Z_u du + e^{-rT} (V_T - F)^+ \right].\end{aligned}\quad (18)$$

The first term in the expectation operator on the right-hand side of (18) is the present value of the dividend payment until time T . From (18), Proposition 2, and the definition of cash flow Z_t , the following proposition is obtained.

Proposition 4 (Equity Value) *Let $A_{a,b}^{\mathbb{Q}}(k, T)$ be the function defined in (15) and*

$$I_a(r, T) := \mathbb{E}^{\mathbb{Q}} \left[\int_0^T e^{-ru} e^{aR_u} du \right]$$

be the deterministic function, whose expression for computation is given by Lemma C.2 in Appendix C. Then, the equity value (18) takes the form

$$E(T) = V_0 \left(\xi \eta I_{\nu}^{\mathbb{Q}}(\bar{r}, T) + e^{-\bar{r}T} \left[A_{\nu,0}^{\mathbb{Q}}(k, T) - A_{0,\nu}^{\mathbb{Q}}(k, T) \right] \right),$$

where $\bar{r} := r + \mu\nu - \zeta$ and $k := \nu^{-1} \log(F/V_0) + (\mu - \zeta\nu^{-1})T$.

In the following, assume a perfect market ($\varepsilon = 1$) for simplicity. If a firm has debt and pays part of its cash flow as dividends to shareholders, the total value of the debt and equity does not necessarily coincide with firm value. That is, $D(T) + E(T) \neq V_0$ when $\eta < 1$. This is because neither debt holders or shareholders do not fully receive the cash flow from the firm. Only when the dividend payout ratio is 100%, the total value of the debt and equity equals firm value. That is, $D(T) + E(T) = V_0$ if and only if $\eta = 1$. When a firm has no debt and 100% dividend payout ratio, the firm is default-free and equity value equals firm value. That is, $\mathbb{P}(V_T < F) = 0$ and $E(T) = V_0$ if $F = 0$ and $\eta = 1$. In this case, with additional conditions $\nu = 1$ and $\zeta = 0$, the equity is entirely identified with the classic Lucas model (Lucas, 1978). For this reason, the model in this paper is a modified version of the Lucas tree model.

Following (13), the second term in the expectation operator on the right-hand side of (18) can be thought of as the price of a *power call option* with exponent ν and strike price F written on aggregate consumption. Therefore, the equity value is nonlinear with respect to the aggregate consumption and is increasing with the volatility of consumption as well as the level of consumption. At-the-money for the option corresponds to the threshold of firm's insolvency. The value of a power call option tends to increase at an accelerated pace when aggregate consumption is near or at-the-money.

The price-dividend ratio defined as

$$\frac{E(T)}{\eta Z_0} = I_{\nu}^{\mathbb{Q}}(\bar{r}, T) + \frac{e^{-\bar{r}T}}{\xi \eta} \left[A_{\nu,0}^{\mathbb{Q}}(k, T) - A_{0,\nu}^{\mathbb{Q}}(k, T) \right],$$

is also an interesting indicator that highly depends on debt ratio F/V_0 . As previously stated, if $F = 0$ and $\eta = 1$, then $E(T) = V_0$ and the model is identical to the classical Lucas model. In this case, the price-dividend ratio is reduced to ξ^{-1} . Although this is a well-known result derived from the Lucas model², it seems unrealistic because the price-dividend ratio is unvarying in terms of firm-specific factors such as the debt ratio. The puzzle called the *stock return predictability* says that future aggregate stock returns are partly predicted by the price-dividend ratio. Concretely, a lower price-dividend ratio forecasts higher stock returns, and vice versa. Since, I am concerned with this anomalously cross-sectional relationship, the question is whether introducing probability weighting and default risk into the model allows for stock return predictability.

²Specifically, in the original Lucas model, dividends are log-normally distributed ($\mathcal{L}(\theta) = \theta$) and identical to the aggregate consumption ($\zeta = \mu$ and $\nu = 1$), and the representative investor has log-utility ($\gamma = 1$). Then, the price-dividend ratio equals δ^{-1} , which is a well-known result because of $\xi = \delta$ by the definition of ξ .

2.8 Equity Risk Premium, Volatility, and Correlation

This subsection presents the definitions of equity risk premium, equity volatility, stock return correlation, and stock index used in the framework. For a single firm, the total amount of the future value for dividend payments until time T and the liquidation value of equity at time T is expressed as

$$\Lambda_T := \int_0^T e^{r(T-u)} \eta Z_u du + (V_T - F)^+. \quad (19)$$

Thence, I define the gross rate of return on the equity realized at time T as $R_T^e := \Lambda_T / E(T)$. Incidentally, the gross rate of return on a risk-free zero-coupon bond with maturity T is defined as $R_T^f := 1/B(T) = e^{rT}$.

2.8.1 Equity Risk Premium and Volatility for Individual Stocks

The equity risk premium on a single stock is defined as

$$\mathbb{E}^\mathbb{P} [R_T^e] - R_T^f = \frac{\mathbb{E}^\mathbb{P} [\Lambda_T]}{E(T)} - e^{rT}, \quad (20)$$

and the equity volatility as

$$\sigma^\mathbb{P} [R_T^e] = \frac{1}{E(T)} \sqrt{\mathbb{E}^\mathbb{P} [(\Lambda_T)^2] - \mathbb{E}^\mathbb{P} [\Lambda_T]^2}, \quad (21)$$

where the computation formulas for the first and second moments of Λ_T are provided in Appendix D.

Similar to the equity value discussed in Section 2.7, the expected future value of total equity payout $\mathbb{E}^\mathbb{P} [\Lambda_T]$ can be thought of as the expected payoff of a power call option written on aggregate consumption. This is the reason why the expected stock returns, equivalent to the equity risk premium, have a nonlinear relation with each other. Firms are expected to have different levels of equity volatility due to different levels of debt ratio, even if they have the same firm value. Note that the equity risk premium and equity volatility are affected by investor probability weighting via equity value $E(T)$, although the superscript on the statistic operators in definitions (20) and (21) takes only objective probability \mathbb{P} at a first glance.

The *excess volatility puzzle* refers to the fact that, from empirical observations, stock prices seem much more volatile than dividend payments. Recall that volatility of dividend payments in the model is identical to asset volatility $\sigma_z = \nu\sigma$, which is also identical to cash flow volatility. Therefore, this puzzle can be formulated by the following inequality:

$$\sigma^\mathbb{P} [R_T^e] \gg \sigma_z.$$

2.8.2 Stock Return Correlation and Stock Index

Now consider a multi-stock market in which there are L firms, indexed by $l = 1, \dots, L$, that might have different firm-specific parameters and debt ratios. Thence, the correlation between the stock returns of firms 1 and 2 is expressed as

$$\text{Corr}^\mathbb{P} [R_T^{e1}, R_T^{e2}] = \frac{\mathbb{E}^\mathbb{P} [\Lambda_{1,T} \Lambda_{2,T}] - \mathbb{E}^\mathbb{P} [\Lambda_{1,T}] \mathbb{E}^\mathbb{P} [\Lambda_{2,T}]}{\sqrt{\mathbb{E}^\mathbb{P} [(\Lambda_{1,T})^2] - \mathbb{E}^\mathbb{P} [\Lambda_{1,T}]^2} \sqrt{\mathbb{E}^\mathbb{P} [(\Lambda_{2,T})^2] - \mathbb{E}^\mathbb{P} [\Lambda_{2,T}]^2}},$$

where $R_T^{e_l}$ and $\Lambda_{l,T}$ denote the stock return and future value of the equity of firm l at time T , respectively. The computation formula for the cross moment between $\Lambda_{1,T}$ and $\Lambda_{2,T}$ is also found in Appendix D. In contrast to equity risk premiums and equity volatility, the stock return correlation is irrelevant to investor risk aversion and probability weighting because the equity values of the two firms affected by investor's characteristics cancel out the correlation. Again, stock returns $R_T^{e_1}$ and $R_T^{e_2}$ are driven by only common factor $W(\tau_t)$, but are nonlinear.

Next, consider the gross rate of return on a portfolio consisting of a weighted sum of every stocks in the market, defined as

$$R_T^m := \sum_{l=1}^L w_l R_T^{e_l}, \quad (22)$$

where w_l is the market portfolio weight of firm l . Hereafter, the portfolio defined above is called the *stock index*. The equity risk premium, volatility, and other characteristics of the stock index are defined in a similar manner to those of individual stocks. In the simulation, I also compute CAPM betas and Jensen's alphas for individual stocks by utilizing the quantities defined above. The *equity premium puzzle* can be summarized as the historical average of excess returns on the stock index being much higher than model-implied equity risk premium $\mathbb{E}^p[R_T^m] - R_T^f$ with reasonable parameters.

3 Calibration and Simulation

This section presents the calibration procedure for the macroeconomic and firm-specific parameters of the model and the simulation results. The simulation addresses widely recognized asset pricing puzzles, while also identifying the anomalous patterns observed among financially distressed stocks. The estimates and graphs are annualized as $T = 1$ unless otherwise stated.

3.1 Consumption Growth Rate Distributions

Consider three stochastic processes for describing the dynamics of objective consumption growth rate R_t . One is the Brownian motion (BM) with mean parameter μ and volatility parameter σ . BM is regarded as the benchmark model in this simulation, because it is normally distributed and business time always coincides with calendar time, that is, $\tau_t = t$ for all $t \geq 0$. The other two models are the variance gamma process (VG) and normal inverse Gaussian process (NIG) with zero skewness. The VG proposed by Madan and Seneta (1990) is a subordinated Brownian motion by the gamma process with parameter κ . The gamma process used as the business time of VG has independently Gamma distributed increments. A sample path of the gamma process is shown in Panel A of Figure 1. VG is a pure jump process without any continuous components, and it has a finite variation at any time, with infinite but relatively low activity of small jumps. On the other hand, the NIG introduced by Barndorff-Nielsen (1997) is a subordinated Brownian motion by the inverse Gaussian process with parameter κ . The increments of the inverse Gaussian process are inverse Gaussian distributed. See Panel B of Figure 1 for a sample path of the inverse Gaussian process. NIG is also a pure jump Lévy process and has infinite variation at any time, with high activity of small jumps. Obviously, both VG and NIG are non-Gaussian processes. Parameter κ generates excess kurtosis of the objective consumption growth rate distribution. Table A.2 exhibits the characteristic exponents and cumulants of the three stochastic processes, BM, VG, and NIG. See also Table A.1 for the models of the business time built in VG and NIG.

For calibrating model parameters, I use annual data on real consumption expenditure in the U.S. from Shiller (2017). The sample period is from 1889 to 2009. The sample mean of the

log consumption growth rates is 2.00%, standard deviation 3.52%, skewness -0.3564, and excess kurtosis 1.1911, as shown in the first row in Panel B of Table 1. These statistics allow calibrating the parameters in the consumption growth rate process: μ , σ , and κ . These parameters are determined by matching the model-generated mean, standard deviation, and excess kurtosis with the corresponding sample statistics. That is, $\mu = 0.0200$, $\sigma = 0.0352$, and $\kappa = 1.1911/3$. Next, I set the time preference parameter $\delta = 0.01$ and risk aversion parameter $\gamma = 4$. This setting follows macro-finance literature. To explore the cause of asset pricing puzzles, I postulate three types of representative investors with different subjective parameters. That is, $\omega = -0.12$, 0, or 0.12, each of which implies the investor's perspective toward consumption is deteriorating, neutral, or improving against a rapidly changing economy, respectively. Panel A of Table 1 summarizes the calibration result for the macroeconomic parameters.

Panel B of Table 1 describes the fundamental statistics of the consumption growth rate distributions. As expected, the benchmark model, BM, does not generate any skewness or excess kurtosis for all distributions. The risk-neutral mean in BM is less than the objective mean. According to Table A.3, the difference between the risk-neutral and objective means is equal to $-\sigma^2\gamma$. This implies it is negative and larger when aggregate consumption is highly volatile or the investor is strongly risk averse. The level of the standard deviation is invariant for any probability measures. These results are also well known in macro-finance literature.

Both VG and NIG accomplish perfect matching for the kurtosis of the objective consumption growth rate distribution with the historical one, as well as the mean and standard deviation. They can thus capture non-normality of the consumption growth rate distribution. On the other hand, objective skewness equals zero and does not match the historical one because of a no skew parameter in the models.

I then turn my attention to the subjective distributions of VG and NIG. The subjective parameter ω in VG controls only standard deviation, while the parameter in NIG distorts kurtosis as well as standard deviation. A negative value of the subjective parameter increases these statistics, and vice versa. However, an impact of the subjective parameter on the subjective distributions is not significant. For example, when $\omega = 0.12$ in VG, the subjective standard deviation equals 3.91%, in contrast with the objective standard deviation of 3.52%. The subjective mean is unchanged, and subjective skewness remains zero. For a more intuitive understanding, Figure 2 plots the probability weighting function defined in (4). The function draws three types of graphs: inverse S-shaped curve, straight line, or S-shaped curve. The inverse S-shaped curve, which is the case of $\omega = -0.12$, is the more frequently observed pattern in past empirical research, particularly in behavioral economics. This shape implies the investor with $\omega = -0.12$ overweighs his/her subjective probability on infrequent events. Conversely, the investor with $\omega = 0.12$ tends to underestimate tail events, because subjective standard deviation and kurtosis are relatively small. Note that the inverse S-shaped curve in Figure 2 is moderate in comparison with the standard illustration of probability weighting functions in textbooks.

Finally, I discuss the risk-neutral distributions in VG and NIG. They are negatively skewed due to investor risk aversion γ and also depend on subjective parameter ω . The level of the risk-neutral mean is smaller than the objective one. An impact of the subjective parameter on the risk-neutral distribution is significantly larger in the case of $\omega = -0.12$ than in the other cases. The risk-neutral standard deviation describes the same pattern as the subjective one in the change of the subjective parameter. That is, a negative value of the subjective parameter increases the risk-neutral standard deviation more sharply. The investor with $\omega = -0.12$ has a more negatively skewed risk-neutral distribution than the investor with $\omega = 0.12$. The risk-neutral kurtosis in VG with $\omega = -0.12$ is relatively larger than in VG with $\omega = 0.12$. However, NIG shows a contrary relation to VG.

Table 1: Consumption Growth Rate Distributions

Panel A reports the macroeconomic parameters used for all the simulation. μ and σ are parameters denoting mean and standard deviation of the objective consumption growth rate distribution. κ is a parameter for VG and NIG generating kurtosis of the distribution. These parameters are calibrated to the historical consumption data in the U.S. from Shiller (2017) over the sample period from 1889 to 2009. δ and γ are time preference and risk aversion parameters, respectively. ω is subjective parameter determining investor's probability weighting. Panel B reports summary statistics of the historical, objective, subjective, and risk-neutral distributions of annual consumption growth rates.

Panel A: Macroeconomic parameters

| μ | σ | κ | δ | γ | ω |
|--------|----------|----------|----------|----------|----------------|
| 0.0200 | 0.0352 | 0.3970 | 0.0100 | 4 | -0.12, 0, 0.12 |

Panel B: Consumption growth rate distributions

| | Probability | Mean | StdDev | Skewness | Kurtosis |
|------------------|--------------|--------|--------|----------|----------|
| Historical | | 0.0200 | 0.0352 | -0.3564 | 1.1911 |
| BM | Objective | 0.0200 | 0.0352 | 0.0000 | 0.0000 |
| | Subjective | 0.0200 | 0.0352 | 0.0000 | 0.0000 |
| | Risk-neutral | 0.0150 | 0.0352 | 0.0000 | 0.0000 |
| VG | Objective | 0.0200 | 0.0352 | 0.0000 | 1.1911 |
| $\omega = -0.12$ | Subjective | 0.0200 | 0.0391 | 0.0000 | 1.1911 |
| | Risk-neutral | 0.0138 | 0.0394 | -0.1852 | 1.2140 |
| $\omega = 0$ | Subjective | 0.0200 | 0.0352 | 0.0000 | 1.1911 |
| | Risk-neutral | 0.0150 | 0.0354 | -0.1668 | 1.2096 |
| $\omega = 0.12$ | Subjective | 0.0200 | 0.0322 | 0.0000 | 1.1911 |
| | Risk-neutral | 0.0158 | 0.0324 | -0.1530 | 1.2067 |
| NIG | Objective | 0.0200 | 0.0352 | 0.0000 | 1.1911 |
| $\omega = -0.12$ | Subjective | 0.0200 | 0.0397 | 0.0000 | 1.5141 |
| | Risk-neutral | 0.0136 | 0.0452 | -0.1813 | 0.9837 |
| $\omega = 0$ | Subjective | 0.0200 | 0.0352 | 0.0000 | 1.1911 |
| | Risk-neutral | 0.0150 | 0.0354 | -0.1671 | 1.2238 |
| $\omega = 0.12$ | Subjective | 0.0200 | 0.0324 | 0.0000 | 1.0135 |
| | Risk-neutral | 0.0157 | 0.0301 | -0.1583 | 1.4248 |

3.2 Interest Rate

Table 2 exhibits historical and model-implied one-year interest rates. The historical one-year interest rate, 2.75%, is the average value of the historical data on real one-year interest rates from Shiller (2017) over the sample period 1871–2012. The model-implied interest rate by BM, 7.99%, is much higher than the historical interest rate. This incompatibility is called the *risk-free rate puzzle*. The steady-state interest rate (SSIR), $\delta + \gamma\mu$, equals 8.98% for all models. The absolute value of the precautionary savings term (PST) in BM, 0.99%, is too small. Recall that PST of BM takes the form $-\sigma^2\gamma^2/2$. To reconcile the risk-free rate puzzle, the absolute value of PST is required to be over 6.00%. This implies that the standard deviation of the consumption growth rate must be at least 8.65% for BM, but this value is unrealistically high.

Next, I discuss the simulation results of VG. The level of the model-implied interest rates depends on the value of subjective parameter ω . Table 2 shows that the PST of VG seems not to be monotonic in ω . Recall that PST is represented as $-\gamma\omega - \mathcal{L}(\gamma^2\sigma^2/2 - \gamma\omega)$. Additionally, Lemma A.2 claims the Laplace exponent of business time, $\mathcal{L}(\theta)$, is an increasing concave function, taking nonnegative values on the positive region. For $\omega = -0.12$, the first term of PST, $-\gamma\omega$, is positive, whereas the second term, $-\mathcal{L}(\gamma^2\sigma^2/2 - \gamma\omega)$, takes a negative value. Summing up the two terms, the value of the corresponding PST equals -6.48%. As a result, the model-implied interest rate in VG is 2.50%, which is close to the historical average of interest rates. Therefore, the investor’s tendency to worry about infrequent events has the possibility to reconcile the risk-free rate puzzle.

As $\omega = 0.12$ under VG, the absolute value of PST is somewhat large, because the first term of PST takes a negative large value despite the positivity of the second term. However, the model-implied interest rate, 4.08%, is still too high in comparison with the historical interest rate. The interest rate in VG with $\omega = 0$ has the same value as in BM. If one calibrates the level of subjective parameter ω such that the model-implied interest rate is fixed at the historical average, 2.75%, the subjective parameter for VG is obtained as $\omega = -0.1174$. The model with calibrated parameter $\omega = -0.1174$ produces analogous simulation results³ to the model with $\omega = -0.12$ shown below.

The model-implied interest rates by NIG are similar to by VG. The interest rate under NIG with $\omega = -0.12$, 1.99%, is smaller than for VG, 2.50%, due to the larger absolute value of the second term of PST in NIG compared to VG. This is owing to the model-specific characteristics of the Laplace exponent $\mathcal{L}(\theta)$. Calibrating the level of the subjective parameter so that the interest rate in NIG is fixed at the historical interest rate, the calibrated parameter is obtained as $\omega = -0.1129$.

As an aspect of the risk-free rate puzzle, Weil (1989) documents that increasing risk aversion leads to a higher interest rate in the standard consumption-based model. Panels A and B of Figure 3 plot interest rates across relative risk aversion γ under VG and NIG, respectively, where the models with $\omega = 0$ cannot circumvent the puzzle. By contrast, when $\omega = -0.12$, the interest rates are decreasing for risk aversion, such that $\gamma > 3$ in both of the two models. This result is evidence that mild overweighting of probability on tail events solves the risk-free rate puzzle.

3.3 Credit Spreads

Consider six credit rating categories: AAA, AA, A, BBB, BB, and B. I postulate representative firms in each rating category, with the corresponding firm-specific profiles listed in Panel A of Table 3. The firms are indexed by $l = \text{AAA}, \dots, \text{B}$. The number of observations, debt ratio, and asset volatility in Panel A are taken from Table 7 in Schaefer and Strebulaev (2008).

³The simulation results in VG with calibrated parameter $\omega = -0.1174$ are available upon request.

Table 2: 1-Year Interest Rate

The table exhibits historical and model-implied one-year interest rates. The historical interest rate is the historical average of real one-year interest rates based on the 1871–2012 sample data from Shiller (2017). SSIR stands for the steady-state interest rate and PST stands for the precautionary savings term.

| | | Interest rate | SSIR | PST |
|------------|------------------|---------------|--------|---------|
| Historical | | 0.0275 | | |
| BM | | 0.0799 | 0.0898 | -0.0099 |
| VG | $\omega = -0.12$ | 0.0250 | 0.0898 | -0.0648 |
| | $\omega = 0$ | 0.0799 | | -0.0099 |
| | $\omega = 0.12$ | 0.0408 | | -0.0490 |
| NIG | $\omega = -0.12$ | 0.0199 | 0.0898 | -0.0699 |
| | $\omega = 0$ | 0.0799 | | -0.0099 |
| | $\omega = 0.12$ | 0.0427 | | -0.0471 |

They estimate these quantities from historical data from 1996 to 2003. The total number of observations is 63,639 and the largest rating category is the A-rated, with 28,044 observations. As shown in Panel A, the debt ratio, defined as the face amount of a firm’s debt divided by the firm value, F/V_0 , is increasing as a result of the deterioration of credit rating. The lowest debt ratio is 10% for the AAA-rated firm, while the highest debt ratio is 66% for the B-rated firm. The values of asset volatility of the firms are approximately the same, ranging between 21% and 23%, except for the B-rated firm, whose asset volatility is 28%. The cumulative 10-year default probabilities in Panel A are based on Moody’s data from 1970 to 1998 as reported by Keenan et al. (2000), which have been used for past credit market analyses (e.g., by Cremers et al., 2008). The quantities stated above are regarded as given sample data for calibration and simulation.

I calibrate firm-specific parameters ν_l and ζ_l for each firm. The value of leverage parameter ν_l defined in (11) is obtained by the relation that asset volatility $\sigma_{z,l}$ is equal to ν_l times standard deviation of consumption growth rate $\sigma = 0.0352$. Next, for a calibrated parameter ν_l , I adjust the level of the cash flow drift term ζ_l in (11), so that the model-implied 10-year default probability in (17) with $T = 10$ matches the sample default probability. The calibration succeeds in fitting all default probabilities together in all the three models, BM, VG, and NIG. The calibration results are exhibited in Panel A of Table 3. The calibrated cash flow drift terms not only for the BBB and BB-rated firms, but also for the AAA-rated firm, are negative. It appears reasonable that the lower rated firms have negative expected growth rates of the cash flow. The reason why the cash flow drift term of the highest rated firm is negative is as follows. The debt ratio of the AAA-rated firm is sufficiently low, at 10%. This implies the default threshold is far away from the current level of firm value. However, because the sample default probability is not zero, 0.77%, the cash flow drift must be negative to match the probability.

Panel B of Table 3 presents historical and model-implied 10-year credit spreads across credit rating categories. Historical credit spreads are taken from Cremers et al. (2008), who estimate them from a 1983–2002 sample of credit spread data used for the Lehman indexes. They range from 66 (AAA) to 548 basis points (B). When computing the model-implied credit spreads, recovery rate is set to 51.3% for all rating categories, $\varepsilon = 0.513$ in (16), which is taken from

Moody’s report by Keenan et al. (2000).

The benchmark model, BM, brings the 10-year credit spread levels from 14 (AAA) to 602 basis points (B). These are underestimated relative to the historical credit spreads, except for the B-rated firm. Particularly, BM generates low credit spreads for investment-grade firms. For example, the 10-year credit spread of the A-rated firm in BM equals 26 basis points, whereas the historical spread is 115 basis points. It is a challenging task to reconcile the model-implied credit spreads with the historically observed ones, especially for high rated firms. This incompatibility is named the *credit spread puzzle* by Amato and Remolana (2003).

I then focus on the simulation results for VG and NIG. The discrepancy between historical and model-implied credit spreads for investment-grade firms are somewhat improved under VG with $\omega = -0.12$. For instance, VG generates the 10-year credit spread of 51 basis points for the A-rated firm. Higher improvement is shown by the NIG with $\omega = -0.12$. The model-implied credit spread of the A-rated firm by NIG equals 56 basis points. The mitigation of the discrepancy is caused by the negative skewness of the risk-neutral consumption growth rate distribution. This implies that the risk-averse investor with $\omega = -0.12$ is very cautious towards the downward jumps of firm values when evaluating corporate bonds. A number of past studies have exogenously incorporated downward jump risk premium into credit market analyses to make the distribution of firm’s growth rate negatively skewed, but they follow only infrequent and large jumps. On the other hand, the proposed model introduces frequent small jumps into the aggregate consumption and generates negative skewness endogenously.

However, the difference is still large. Several empirical studies have documented the presence of unrelated factors to firms’ default in credit spreads. For example, Elton et al. (2001) point out the effect of a state tax on corporate bond coupons, which is not levied on Treasury bond coupons. They demonstrate that the tax effect is increasing credit spreads for investment-grade firms. Another plausible factor is the liquidity effect. Generally, corporate bond markets are less liquid than government ones. Chen et al. (2007) conclude that liquidity is priced in yield spreads of corporate bonds and explains as much as half of the cross-sectional variation in credit spread changes. However, such an effect is outside the scope of this paper. In contrast to investment-grade firms, the model-implied 10-year credit spreads for the B-rated firm are overestimated relative to historical spreads. This is also controversial.

3.4 Non-Distressed Stocks

3.4.1 Equity Risk Premium and Volatility across Rating Categories

Table 4 describes the equity risk premiums and equity volatility of the stock index, the representative firms in each credit rating category, and a virtual non-defaultable firm. The first row exhibits market portfolio weights, defined as the ratio of a firm’s equity value to the total stock market value, calculated based on Table 7 in Schaefer and Strebulaev (2008). The historical equity premium of 5.2% and equity volatility of 18.2% on the stock index are estimated from annual returns on S&P500, as obtained from Shiller’s (2017) data from 1871 to 2011. The historical equity volatility of the representative firms ranging from 25% (AAA) to 61% (B) are taken from Schaefer and Strebulaev (2008), whereas their equity risk premiums are unfortunately not available. For the simulation, I set the dividend payout ratio equal to 20% for any firms, that is, $\eta_l = 0.20$ for all l . The model-implied equity risk premiums and equity volatility of each firm are calculated by (20) and (21), while those of the stock index are based on (22). The non-defaultable firm is a virtual firm with debt ratio zero and assumed to have the same asset volatility, 22%, as the AAA-rated firm and a cash flow drift term of 2% equal to the average growth rate of historical consumption.

I first discuss the benchmark model, BM. The equity risk premium of the non-defaultable

Table 3: Firm-Specific Parameters and Credit Spreads

In Panel A, *Number of observations*, *Debt ratio*, and *Asset volatility* are taken from Table 7 in Schaefer and Strebulaev (2008), and the *10-year default probability* for each rating category is based on Moody's report by Keenan et al. (2000) from 1970 to 1998. Firm-specific parameters ν_l and ζ_l for rated firms indexed by $l = \text{AAA}, \dots, \text{B}$ denote leverage and cash flow drift parameters, respectively. These parameters are in turn calibrated to the asset volatility and 10-year default probability. Panel B reports historical and model-implied 10-year credit spreads of corporate bonds, whose unit are basis points. The historical credit spreads are taken from Cremers et al. (2008), who estimate them from the 1983–2002 sample data used for the Lehman indexes.

| Panel A: Firm-specific parameters | | | | | | | |
|-----------------------------------|--------|--------|-------|--------|--------|--------|--------|
| | Total | AAA | AA | A | BBB | BB | B |
| Number of observations | 63,639 | 1,318 | 6,713 | 28,044 | 22,183 | 4,548 | 833 |
| Debt ratio | 0.34 | 0.10 | 0.21 | 0.32 | 0.37 | 0.50 | 0.66 |
| Asset volatility | 0.22 | 0.22 | 0.22 | 0.21 | 0.22 | 0.23 | 0.28 |
| 10-year default probability | | 0.77% | 0.99% | 1.55% | 4.39% | 20.63% | 43.91% |
| Firm-specific parameters | | | | | | | |
| ν_l | | 6.256 | 6.256 | 5.971 | 6.256 | 6.540 | 7.962 |
| ζ_l | BM | -0.062 | 0.006 | 0.029 | 0.019 | -0.010 | -0.028 |
| | VG | -0.059 | 0.008 | 0.030 | 0.019 | -0.010 | -0.028 |
| | NIG | -0.059 | 0.008 | 0.030 | 0.019 | -0.010 | -0.028 |

| Panel B: 10-year credit spreads (basis points) | | | | | | | |
|--|------------------|-----|----|-----|-----|-----|-----|
| | | AAA | AA | A | BBB | BB | B |
| Historical | | 66 | 92 | 115 | 171 | 332 | 548 |
| BM | | 14 | 18 | 26 | 66 | 268 | 602 |
| VG | $\omega = -0.12$ | 32 | 38 | 51 | 106 | 331 | 662 |
| | $\omega = 0$ | 17 | 21 | 30 | 71 | 275 | 603 |
| | $\omega = 0.12$ | 9 | 12 | 18 | 49 | 233 | 560 |
| NIG | $\omega = -0.12$ | 36 | 42 | 56 | 112 | 339 | 669 |
| | $\omega = 0$ | 17 | 21 | 30 | 72 | 275 | 603 |
| | $\omega = 0.12$ | 9 | 12 | 19 | 51 | 237 | 564 |

firm equals 3.4%. If all firms were non-defaultable, which is standard in past asset pricing literature, the equity premium of the stock index would be 3.4%. This hypothetical situation can be identified with the classical Lucas tree model, and the model-implied equity premium seems too low relative to the historical average of excess returns on S&P500, 5.2%. The discrepancy stated above is called the *equity premium puzzle*, as reported by Mehra and Prescott (1985).

As shown in the simulation result of BM, by incorporating default risk into an asset pricing model, the model generates higher equity risk premiums for individual firms. The level of the equity premiums ranges from 3.7% (AAA) to 11.8% (B). The worse is the credit rating, the more increases the equity risk premium. This result essentially embodies MM proposition II (Modigliani & Miller, 1958) that the expected return on equity increases as the debt to equity ratio increases. As a result of the higher individual equity premiums, the equity risk premium of the stock index being capitalization-weighted equals 5.0%, which is close to the level of the historical equity risk premium, 5.2% and higher than the equity risk premium of the non-defaultable firm, 3.4%, the average growth rate of historical consumption, 2.0%, and the weighted average of the cash flow drifts of the representative firms, 1.8%. This result indicates default risk has the potential to reconcile the equity premium puzzle.

Going back to the non-defaultable firm in BM, the equity volatility of the non-defaultable firm is 24.7%, while the asset volatility is set to 22% as stated above. Recall the asset volatility of a firm is identical to the volatility of its dividend payment in the framework. Shiller (1981) documents that stock price fluctuations observed on markets are excessive relative to dividend payment fluctuations. This is called the *excess volatility puzzle*. The non-defaultable stock in the simulation ends up reproducing this puzzle. In fact, the level of the equity volatility on the non-defaultable firm is close to asset volatility. Furthermore, if the dividend payout ratio η were 100%, they would take the same percentage.

Incorporating default risk into the benchmark model causes the different levels of equity volatility across credit rating categories to range from 27.3% (AAA) to 89.3% (B). Table 4 shows that model-implied equity volatility is increasing with the deterioration of the credit rating consistent with the historical equity volatility. As expected, equity risk premiums are higher for higher equity volatility, and the level of the model-implied equity volatility is much larger than the asset volatility listed in Panel A of Table 3. The financial leverage effect naturally described in a structural credit model thus makes stock prices highly volatile. Therefore, introducing default risk into the model can reconcile the excess volatility puzzle.

In BM, the equity volatility of each firm has nearly the same level as the historical equity volatility, whereas the stock index volatility, 36.3%, is obviously higher than the historical volatility of S&P500, 18.2%. This inconsistency is attributed to two reasons. One is that the economy in this framework is driven by only one common factor, $W(\tau_t)$, and all firms do not have any idiosyncratic driving factors for simplicity. As a result, the individual stock returns generated by the model are highly correlated with each other. Model-implied correlations among them take values above 0.992. Due to the extremely high correlations, the diversification effect on an equity portfolio, which could be expected to reduce its volatility, scarcely works. The other reason is the difference of sample firms between S&P500 and this examination. The number of observations in the simulation is 63,639, as shown in Panel A of Table 3, while S&P500 is, of course, based on the market capitalizations of 500 large companies listed on NYSE or NASDAQ.

Next, I discuss the simulation results under VG and NIG. Table 4 reveals that the impact of subjective parameter ω on equity risk premiums and equity volatility is not significant. For example, the non-defaultable firm has an equity risk premium between 3.2% and 3.4% and equity volatility between 23.9% and 25.2%. Roughly, they are the same levels as those of the non-defaultable firm in BM. Therefore, even if the investor has probability weighting when

evaluating stock prices, the stock returns are far from the observational evidence.

Focusing on credit rated firms in VG and NIG, the level of the equity risk premiums ranges from 3.5% (AAA in VG or NIG with $\omega = 0.12$) to 12.3% (B in VG or NIG with $\omega = 0.12$), and the level of the equity volatility ranges from 26.5% (AAA in NIG with $\omega = -0.12$) to 91.5% (B in NIG with $\omega = 0$). Consequently, the equity risk premium of the stock index takes a value between 4.7% and 5.0%, while stock index volatility ranges from 35.4% to 37.0%. These estimates depict similar patterns to those in the benchmark model, BM. This result indicates that an asset pricing model with default risk separate from investor's probability weighting has the potential to resolve the equity premium and excess volatility puzzles. It is worth recalling that the investor with negative subjective parameter cancels out the risk-free rate puzzle. In conclusion, the combination of default risk and probability weighting on rare events is expected to provide an answer to the widely recognized asset pricing puzzles in financial economics. However, subjective parameter ω appears to hardly affect stock returns, but this intuition is not correct in financially distressed stocks (see Section 3.5 for details).

Table 4: Equity Risk Premium and Volatility across Rating Categories

The table shows equity risk premiums and equity volatility of the stock index, representative firms in each credit rating category, and a non-defaultable firm. The historical equity premium of the stock index is based on the average excess return on S&P500 computed from the 1871–2011 sample data in Shiller (2017), while the historical equity volatility across rating categories are taken from Table 7 in Schaefer and Strebulaev (2008). *Market value weights*, which are based on the data in Table 7 in Schaefer and Strebulaev (2008) and used for computing the model-implied stock index risk premiums and volatility, are defined as the ratios of firms' equity values to the total stock market value. The non-defaultable firm is assumed to have no debt with a cash flow drift of 2%, asset volatility of 22%, and dividend payout ratio of 20%.

| | Index | AAA | AA | A | BBB | BB | B | Non-defaultable |
|---------------------|----------------|-------|-------|-------|-------|-------|-------|-----------------|
| Market value weight | 1.000 | 0.028 | 0.126 | 0.453 | 0.332 | 0.054 | 0.007 | |
| Historical | Equity premium | | | | | | | |
| | Equity vol | 0.25 | 0.29 | 0.31 | 0.33 | 0.42 | 0.61 | |
| BM | Equity premium | 0.037 | 0.043 | 0.047 | 0.053 | 0.070 | 0.118 | 0.034 |
| | Equity vol | 0.273 | 0.311 | 0.341 | 0.386 | 0.514 | 0.893 | 0.247 |
| VG | Equity premium | 0.038 | 0.043 | 0.048 | 0.054 | 0.070 | 0.100 | 0.034 |
| $\omega = -0.12$ | Equity vol | 0.266 | 0.303 | 0.332 | 0.378 | 0.505 | 0.879 | 0.240 |
| $\omega = 0$ | Equity premium | 0.038 | 0.043 | 0.047 | 0.053 | 0.070 | 0.116 | 0.034 |
| | Equity vol | 0.278 | 0.317 | 0.346 | 0.394 | 0.523 | 0.913 | 0.252 |
| $\omega = 0.12$ | Equity premium | 0.035 | 0.040 | 0.044 | 0.050 | 0.067 | 0.123 | 0.032 |
| | Equity vol | 0.269 | 0.306 | 0.335 | 0.381 | 0.509 | 0.903 | 0.243 |
| NIG | Equity premium | 0.038 | 0.043 | 0.048 | 0.054 | 0.069 | 0.097 | 0.034 |
| $\omega = -0.12$ | Equity vol | 0.265 | 0.302 | 0.331 | 0.377 | 0.503 | 0.877 | 0.239 |
| $\omega = 0$ | Equity premium | 0.038 | 0.043 | 0.047 | 0.053 | 0.070 | 0.117 | 0.034 |
| | Equity vol | 0.278 | 0.317 | 0.347 | 0.394 | 0.523 | 0.915 | 0.252 |
| $\omega = 0.12$ | Equity premium | 0.035 | 0.040 | 0.044 | 0.050 | 0.067 | 0.123 | 0.032 |
| | Equity vol | 0.269 | 0.307 | 0.336 | 0.382 | 0.510 | 0.906 | 0.243 |

3.4.2 Simulation Results across Firm-Specific Factors

Here, I simulate the cross-sectional relationship between stock returns and firm-specific factors. Recall that the model has four firm-specific factors: cash flow drift term ζ , dividend payout ratio η , asset volatility σ_z , and debt ratio F/V_0 . Assume a benchmark firm with $\zeta = 1\%$, $\eta = 20\%$, $\sigma_z = 22\%$, and $F/V_0 = 30\%$ in this simulation. Moreover, suppose an economy in which the consumption growth rate process follows VG and the representative investor overweighs his/her probability on rare events ($\omega = -0.12$). Then, I compute five estimates relevant to stock return performance measurement: equity risk premium, equity volatility, logarithm of price-dividend ratio, CAPM beta, and Jensen's alpha across the changes of each firm-specific factor.

Table 5 describes the simulation results. First, Panels A and B exhibit the estimates in terms of the variation of cash flow drift term and dividend payout ratio, respectively. The impact of these factors on the equity risk premiums and volatility appears to be subtle, despite varying the log price-dividend ratios. The levels of the CAPM betas range from 0.957 to 0.966 in Panel A and from 0.955 to 0.964 in Panel B, while Jensen's alphas in both Panels A and B are approximately zero. However, the conclusion that cash flow drift terms are entirely irrelevant to stock returns is incorrect. As shown in Section 3.5, a severe deterioration of the expected cash flow growth rate significantly affects stock returns.

Second, I discuss the simulation result in Panel C, which shows the impact of asset volatility on the stock return performance measurement. As expected, the higher asset volatility is, the larger are both equity premium and equity volatility. This result is relevant to the CAPM beta generated by the model increasing under asset volatility. The level of the equity premiums ranges from 3.3% to 6.2%, while the CAPM beta ranges from 0.594 to 1.351. Jensen's alpha is slightly decreasing under asset volatility.

Finally, Panel D exhibits the estimates across debt ratio. The higher the debt ratio is, the larger both equity premium and equity volatility are. This is the financial leverage effect claimed by the MM proposition. Similarly to Panel C, stocks with higher debt ratio have higher CAPM betas, whereas Jensen's alphas are always around zero. Precisely, the equity premium takes a value between 3.8% and 6.7% and the level of the CAPM betas ranges from 0.749 to 1.338.

In conclusion, from Table 5, increasing asset volatility or debt ratio earns higher expected stock returns and equity volatility. This is because higher market risk is embedded in such stocks. Moreover, the stock returns seem to not involve other risk components such as small capital or value factors in the Fama-French three-factor model (Fama & French, 1993). Basically, the simulation results are in line with the seminal Modigliani and Miller theory that as the proportion of debt in the firm's capital structure increases, its equity return to the shareholder increases. Additionally, they reveal that the price-dividend ratio is not necessarily an immediate clue to predict future stock returns in the framework. However, stock return predictability still remains as a puzzle.

3.5 Distressed Stocks

3.5.1 Observational Evidence

Dichev (1998), Griffin and Lemmon (2002), Campbell et al. (2008), Avramov et al (2009), and others recognize the existence of the cross-sectional relationship that average excess returns on financially distressed stocks are decreasing with the deterioration of default risk. This observational evidence seems to be anomalous, because it means that investors pay premiums for stocks with higher default possibility. Griffin and Lemmon (2002) conclude that such stocks are mispriced, and Avramov et al. (2009) think of this relation as a type of asset pricing puzzle. Campbell et al. (2008) argue three possible reconciliations for the anomalous underperformance

Table 5: Stock Returns across Firm-Specific Factors

The table reports the simulation results, including equity risk premiums, equity volatility, log price-dividend ratios, CAPM betas, and Jensen's alphas. This simulation postulates a benchmark firm that has a cash flow drift term of 1%, asset volatility of 22%, debt ratio of 30%, and dividend payout ratio of 20%. Panels A–D exhibit the results for the firm with changing in level of each firm-specific factor.

| Panel A: Cash flow drift term | | | | | |
|-------------------------------|-------|-------|-------|-------|-------|
| Cash flow drift term | -0.01 | 0.00 | 0.01 | 0.02 | 0.03 |
| Equity premium | 0.049 | 0.049 | 0.048 | 0.048 | 0.048 |
| Equity vol | 0.343 | 0.342 | 0.341 | 0.340 | 0.340 |
| Log PD ratio | 4.351 | 4.625 | 4.992 | 5.562 | 6.948 |
| CAPM beta | 0.966 | 0.964 | 0.962 | 0.959 | 0.957 |
| Jensen's alpha | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |

| Panel B: Dividend payout ratio | | | | | |
|--------------------------------|-------|-------|-------|-------|-------|
| Dividend payout ratio | 0.10 | 0.20 | 0.30 | 0.40 | 0.50 |
| Equity premium | 0.049 | 0.048 | 0.048 | 0.048 | 0.048 |
| Equity vol | 0.342 | 0.341 | 0.340 | 0.340 | 0.339 |
| Log PD ratio | 5.682 | 4.992 | 4.590 | 4.305 | 4.085 |
| CAPM beta | 0.964 | 0.962 | 0.959 | 0.957 | 0.955 |
| Jensen's alpha | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |

| Panel C: Asset volatility | | | | | |
|---------------------------|-------|-------|-------|--------|--------|
| Asset volatility | 0.14 | 0.18 | 0.22 | 0.26 | 0.30 |
| Equity premium | 0.033 | 0.041 | 0.048 | 0.056 | 0.062 |
| Equity vol | 0.211 | 0.275 | 0.341 | 0.410 | 0.481 |
| Log PD ratio | 4.834 | 4.871 | 4.992 | 5.234 | 5.712 |
| CAPM beta | 0.594 | 0.775 | 0.962 | 1.154 | 1.351 |
| Jansen's alpha | 0.003 | 0.002 | 0.000 | -0.002 | -0.006 |

| Panel D: Debt ratio | | | | | |
|---------------------|-------|-------|-------|-------|--------|
| Debt ratio | 0.10 | 0.20 | 0.30 | 0.40 | 0.50 |
| Equity premium | 0.038 | 0.042 | 0.048 | 0.056 | 0.067 |
| Equity vol | 0.266 | 0.299 | 0.341 | 0.397 | 0.475 |
| Log PD ratio | 5.242 | 5.125 | 4.992 | 4.840 | 4.660 |
| CAPM beta | 0.749 | 0.842 | 0.962 | 1.120 | 1.338 |
| Jensen's alpha | 0.000 | 0.000 | 0.000 | 0.000 | -0.001 |

of distressed stocks: (i) unexpected events in observed period, (ii) valuation errors by irrational investors, and (iii) rational valuation to hold them despite low equity premiums. However, Campbell et al. (2008) do not construct any asset pricing models to explain the anomaly. To the best of my knowledge, Garlappi and Yan's (2011) is only the study to provide a theoretical reconciliation for it. Specifically, Garlappi and Yan (2011) develop an equity valuation model based on a partial equilibrium approach, which is explicitly incorporated with financial leverage. They prove that the presence of potential shareholder recovery upon financial distress causes a cross-sectional anomaly. This section thus provides another theoretical explanation based on the consumption-based asset pricing approach.

Another topic about distressed stocks is correlation structure. Financially distressed stocks are known to be highly volatile and less correlated with a market portfolio. However, as noted by Campbell et al. (2008), the high volatility cannot fully diversify at a portfolio level constructed by distressed stocks, and there exists a degree of covariation in them. This empirical evidence indicates the high volatility of distressed stocks is caused not only by idiosyncratic firm-level risk factors, but also by a latent common factor driving them. Campbell et al. (2008) and Avramov et al. (2009) also report that distressed stocks have high CAPM betas and negative Jensen's alphas. The analysis in this paper explores the possibility to reproduce this apparently inconsistent phenomenon.

In sum, the purpose of this subsection is to offer a new perspective for understanding the observational evidence mentioned above. The model is explicitly incorporated with the default risk of individual firms and probability weighting of the representative investor. This modeling is expected to provide a consistent and unified reconciliation to the anomalous patterns in distressed stocks.

3.5.2 Simulation Results

I postulate a benchmark firm in financial distress as follows. Cash flow drift term ζ is set to -0.2. That is, the cash flow generated by the firm is expected to decline 20% per year. Asset volatility σ_z is set to 21.12% ($\nu = 6$) and debt ratio F/V_0 to 100%. The latter assumption means that the debt level is at insolvency and the intrinsic value of the equity is worthless, while it might have a positive time value for the equity. The benchmark distressed firm does not pay dividends to shareholders at any time, $\eta = 0$.

Panel A of Table 6 describes the simulation results for the distressed stock across debt ratios. In this simulation, the higher the debt ratio of the firm is, the worse the credit risk. First, I focus on the simulation result in benchmark model BM. The equity risk premium ranging from 20.5% to 47.7% is extremely high and increasing with the change of debt ratio. This result is incompatible with the empirical observation above, whereas it accords with the standard theory of asset pricing in that assets with relatively high risk are expected to earn high average returns.

Second, I discuss the simulation results in VG and NIG. Panel A shows that equity premium on the distressed stock are decreasing with the change of debt ratio when the investor has a negative value of subjective parameter, $\omega = -0.12$. Therefore, the overweighting probability on tail events induces the anomalously cross-sectional relationship reported in past empirical analyses. Surprisingly, the models predict negative equity premiums for a firm falling into excessive liabilities. Indeed, VG and NIG generate negative premiums ranging from -0.2% (VG with $\omega = -0.12$ and debt ratio of 1.2) to -8.8% (NIG with $\omega = -0.12$ and debt ratio of 1.2). By contrast, when $\omega = 0$ or 0.12, the cross-sections between equity premiums and debt ratios have a positive relationship. Neither underweighting probability on rare events nor non-biased subjective probability reproduce the anomalous pattern.

It is worth noting that the correlation between returns on the distressed stock and the stock

index, which ranges between 0.924 and 0.439 in BM, is weakening with the increasing debt ratio. On the other hand, correlations among distressed stocks are close to one. For instance, the correlation between the stock returns of two firms with debt ratios of 1.0 and 1.1 equals 0.935. The simulation result is thus consistent with the observational evidence stated by Campbell et al. (2008), which is caused by optionality of the future values of equity. Recall that an equity value can be regarded as the price of a power call option written on aggregate consumption. A call option price is an increasing and convex function of the underlying asset price. Particularly, distressed stocks can be thought of as near-the-money call options. Distressed stocks staying near a position on the convex curve move together via consumption fluctuations. That is, they have *near option gammas*. Consequently, they are highly correlated with each other. By contrast, stocks issued by investment-grade firms can be thought of as deep in-the-money call options. Because of nonlinearity, distressed stocks are weakly correlated with the stock index, in which investment-grade firms account for the majority. That is, a distressed stock has a *different option gamma* from the stock index. Seemingly paradoxical observations of the correlation structure are compatible in this model.

Panel A also shows that the CAPM betas of distressed stocks are high despite weak correlations with the stock index. Moreover, Jensen's alphas take negative values for $\omega = -0.12$ or 0, while they are otherwise positive. The simulation results are consistent with observational evidence. In the framework, applying CAPM to distressed stocks means a forcible linear regression of their returns against the stock index despite a nonlinear relationship. This is the reason why CAPM betas are high and Jensen's alphas are nonzero.

Panel B presents the simulation results for the distressed stock across cash flow drift. In this simulation, the more negative the cash flow drift term is, the worse the credit risk. As a rule, the simulation results in Panel B are analogous to those in Panel A. Both VG and NIG with $\omega = -0.12$ reconcile the anomalous patterns observed in distressed stocks. The two models reproduce the positive cross-sectional relation between cash flow drifts and equity premiums. They also predict negative Jensen's alphas. All results display high volatility and large CAPM, as well as a correlation structure consistent with the empirical observation. By contrast, when $\omega = 0$ or 0.12, the equity risk premium is increasing in deterioration of the cash flow drift. Jensen's alphas take positive values for $\omega = 0.12$.

In conclusion, the simulation shows that, if the representative investor had more probability weighting on infrequent events, the seemingly anomalous patterns in financially distressed stocks might be rational for him/her. That is, distressed stocks are *not* mispriced. The simulation results strongly support Campbell et al.'s (2008) third possible reconciliation for the distressed equity premium puzzle: (iii) rational valuation to hold financially distressed stocks despite low equity premiums. The model also succeeds in offering a theoretical explanation for the correlation structure among distressed stocks.

3.5.3 Discussion

In the following, I discuss why the model can generate the anomalous cross-sectional relationship between default risk and equity risk premium. To theoretically understand the relationship, there are two key points: one is the shape of the projection of the pricing kernel onto the consumption growth rate and the other the interpretation of an equity value with default risk as a power call option written on consumption. First, the discussion focuses on the projection of the pricing kernel onto the consumption growth rate, defined as

$$\frac{dQ}{dP}(R_t < x) = \frac{dP^S}{dP} \times \frac{dQ}{dP^S}(R_t < x), \quad (23)$$

which is the ratio of the risk-neutral density function of the log consumption growth rate to the objective density function. It is equivalent to the probability weighting function times the ratio of the risk-neutral density function to the subjective density function. The projection of the pricing kernel (23) can be easily computed by Lévy's inversion formula of the characteristic function (6), with exponents (9) and (7). Panels A and B in Figure 4 plot the projections under VG and NIG, respectively. When $\omega = 0$, they are monotonically decreasing with the consumption growth rate. This shape accords with standard asset pricing theory, for instance, Lucas (1978). However, several empirical studies, such as Ait-Sahalia and Lo (1998), Jackwerth (2000), and Rosenberg and Engle (2002), have demonstrated that pricing kernels are not monotonically decreasing, but have some increasing regions. Furthermore, Bakshi et al. (2010) assert that empirical pricing kernels observed on the U.S. stock market are U-shaped, being compatible with $\omega = -0.12$ in Figure 4. They also develop a general theory of U-shaped pricing kernels linked with contingent claims. I cite the following theorem from Bakshi et al. (2010), which is crucial to understand the simulation results.

Theorem 1 (Bakshi et al., 2010) *If the economy supports a U-shaped pricing kernel, then the following statements are true:*

1. *Expected returns of call options with strike prices above a certain level are decreasing in the level of the strike prices.*
2. *There exists a strike price K such that call options with strikes higher than K have negative expected returns.*
3. *The steeper the slope of the U-shaped pricing kernel in the increasing region, the smaller are the expected returns of call options.*

Under this framework, stocks issued by financially distressed firms are identical to near-the-money or out-of-the-money call options written on ν -powered consumption. Following Theorem 1, if the pricing kernel is U-shaped, then the equity risk premiums of firms incurring debt above a certain level are decreasing with the debt level, and could be negative in the range of deep out-of-the-money. Moreover, Polkovnichenko and Zhao (2013) reveal the relation between probability weighting functions and the slope of pricing kernels. The simulation results are consistent with Polkovnichenko and Zhao's (2013). However, all past studies treating U-shaped pricing kernels have not addressed distressed stock markets, but option markets.

Next, I argue an economic interpretation of low equity risk premiums on distressed stocks. Concretely, I consider why investors are overweighing probability on rare events, being willing to purchase distressed stocks despite low expected returns. Barberis and Huang (2008) have already answered an analogous question. They insist that such investors are willing to buy a positively skewed security, for example, an IPO stock, although the security is expected to earn a slightly positive or negative excess return. They call such a security a *lottery*. One can also interpret distressed stocks as lotteries, because the payoff distributions of the distressed stocks are positively skewed. Consequently, the investors with a pessimistic perspective on consumption for a rapidly changing economy are willing to hold a distressed stock as lottery and believe a rare event that the firm has dramatically improved or if a bail-out occurs. Recall that the probability weighting with $\omega = -0.12$ is not intensive, but moderate. In conclusion, the aggregation of investors having *mild* overweighing of probability on infrequent events could resolve the anomalous cross-sectional pattern.

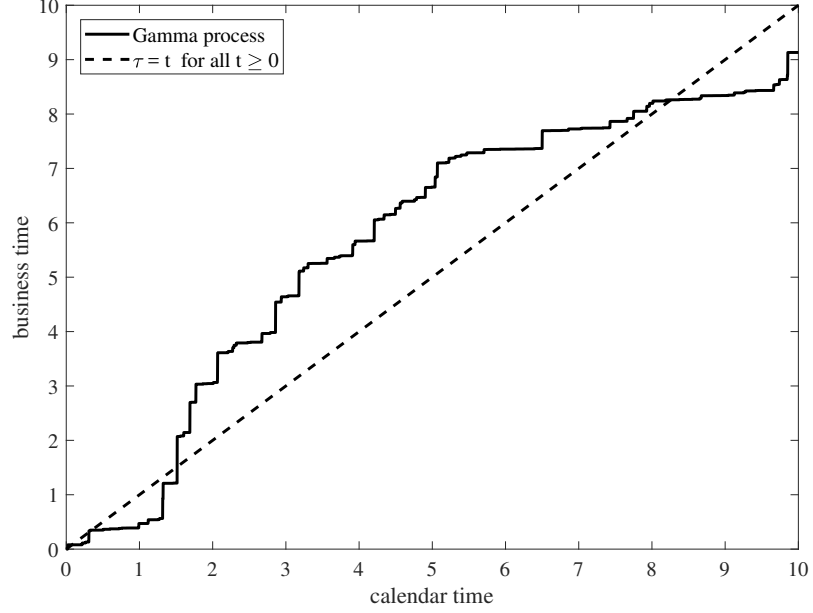
Table 6: Equity Risk Premium and Volatility on Distressed Stocks

The table reports the simulation results for financially distressed stocks, which provide correlations with the stock index, equity risk premiums, equity volatility, CAPM betas, and Jensen's alphas. This simulation postulates a benchmark firm falling into financial distress, with cash flow drift term of -20%, asset volatility of 21.12%, debt ratio of 100%, and no dividend payment. Panels A and B exhibit the results for the firm with changing levels of debt ratio and cash flow drift term, respectively.

| | | Panel A: Debt ratio | | | | |
|------------------|----------------|---------------------|--------|--------|--------|--------|
| Debt ratio | | 0.8 | 0.9 | 1.0 | 1.1 | 1.2 |
| BM | Correlation | 0.924 | 0.832 | 0.708 | 0.572 | 0.439 |
| | Equity premium | 0.205 | 0.276 | 0.342 | 0.409 | 0.477 |
| | Equity vol | 1.867 | 2.864 | 4.445 | 7.067 | 11.424 |
| | CAPM beta | 4.746 | 6.555 | 8.667 | 11.114 | 13.797 |
| | Jensen's alpha | -0.031 | -0.050 | -0.089 | -0.144 | -0.210 |
| VG | Correlation | 0.921 | 0.833 | 0.726 | 0.619 | 0.520 |
| $\omega = -0.12$ | Equity premium | 0.112 | 0.088 | 0.058 | 0.027 | -0.002 |
| | Equity vol | 1.799 | 2.705 | 3.948 | 5.621 | 7.809 |
| | CAPM beta | 4.669 | 6.346 | 8.081 | 9.804 | 11.448 |
| | Jensen's alpha | -0.123 | -0.231 | -0.349 | -0.466 | -0.578 |
| $\omega = 0$ | Equity premium | 0.216 | 0.306 | 0.393 | 0.479 | 0.565 |
| | Equity vol | 2.055 | 3.376 | 5.379 | 8.341 | 12.575 |
| | CAPM beta | 5.116 | 7.601 | 10.563 | 13.958 | 17.686 |
| | Jensen's alpha | -0.040 | -0.073 | -0.135 | -0.218 | -0.318 |
| $\omega = 0.12$ | Equity premium | 0.319 | 0.501 | 0.735 | 0.996 | 1.275 |
| | Equity vol | 2.152 | 3.749 | 6.475 | 10.881 | 17.677 |
| | CAPM beta | 5.532 | 8.715 | 13.130 | 18.802 | 25.672 |
| | Jensen's alpha | 0.060 | 0.094 | 0.121 | 0.118 | 0.075 |
| NIG | Correlation | 0.922 | 0.832 | 0.722 | 0.612 | 0.513 |
| $\omega = -0.12$ | Equity premium | 0.105 | 0.075 | 0.025 | -0.031 | -0.088 |
| | Equity vol | 1.767 | 2.642 | 3.838 | 5.410 | 7.370 |
| | CAPM beta | 4.607 | 6.220 | 7.841 | 9.367 | 10.698 |
| | Jensen's alpha | -0.128 | -0.239 | -0.370 | -0.503 | -0.628 |
| $\omega = 0$ | Equity premium | 0.214 | 0.303 | 0.391 | 0.482 | 0.572 |
| | Equity vol | 2.039 | 3.345 | 5.413 | 8.555 | 13.091 |
| | CAPM beta | 5.081 | 7.527 | 10.569 | 14.157 | 18.161 |
| | Jensen's alpha | -0.039 | -0.073 | -0.137 | -0.225 | -0.335 |
| $\omega = 0.12$ | Equity premium | 0.316 | 0.472 | 0.710 | 0.996 | 1.322 |
| | Equity vol | 2.136 | 3.657 | 6.436 | 11.150 | 18.710 |
| | CAPM beta | 5.485 | 8.482 | 12.955 | 19.022 | 26.759 |
| | Jensen's alpha | 0.058 | 0.074 | 0.102 | 0.103 | 0.066 |

| Panel B: Cash flow drift term | | | | | | |
|-------------------------------|----------------|--------|--------|--------|--------|--------|
| Cash flow drift term | | 0.0 | -0.1 | -0.2 | -0.3 | -0.4 |
| BM | Correlation | 0.923 | 0.826 | 0.708 | 0.565 | 0.412 |
| | Equity premium | 0.231 | 0.278 | 0.342 | 0.414 | 0.497 |
| | Equity vol | 2.080 | 2.920 | 4.445 | 7.237 | 12.778 |
| | CAPM beta | 5.283 | 6.637 | 8.667 | 11.244 | 14.482 |
| | Jensen's alpha | -0.031 | -0.052 | -0.089 | -0.145 | -0.224 |
| VG | Correlation | 0.920 | 0.827 | 0.726 | 0.613 | 0.501 |
| $\omega = -0.12$ | Equity premium | 0.114 | 0.086 | 0.058 | 0.025 | -0.009 |
| | Equity vol | 1.988 | 2.754 | 3.948 | 5.714 | 8.362 |
| | CAPM beta | 5.158 | 6.422 | 8.081 | 9.879 | 11.797 |
| | Jensen's alpha | -0.145 | -0.237 | -0.349 | -0.471 | -0.602 |
| $\omega = 0$ | Equity premium | 0.246 | 0.310 | 0.393 | 0.485 | 0.586 |
| | Equity vol | 2.319 | 3.451 | 5.379 | 8.527 | 13.728 |
| | CAPM beta | 5.773 | 7.721 | 10.563 | 14.143 | 18.581 |
| | Jensen's alpha | -0.042 | -0.076 | -0.135 | -0.221 | -0.342 |
| $\omega = 0.12$ | Equity premium | 0.345 | 0.511 | 0.735 | 1.010 | 1.342 |
| | Equity vol | 2.419 | 3.846 | 6.475 | 11.157 | 19.602 |
| | CAPM beta | 6.217 | 8.885 | 13.130 | 19.110 | 27.397 |
| | Jensen's alpha | 0.054 | 0.096 | 0.121 | 0.117 | 0.062 |
| NIG | Correlation | 0.921 | 0.827 | 0.722 | 0.607 | 0.494 |
| $\omega = -0.12$ | Equity premium | 0.107 | 0.072 | 0.025 | -0.034 | -0.099 |
| | Equity vol | 1.950 | 2.688 | 3.838 | 5.497 | 7.865 |
| | CAPM beta | 5.083 | 6.288 | 7.841 | 9.431 | 10.989 |
| | Jensen's alpha | -0.150 | -0.245 | -0.370 | -0.509 | -0.653 |
| $\omega = 0$ | Equity premium | 0.244 | 0.307 | 0.391 | 0.488 | 0.595 |
| | Equity vol | 2.297 | 3.421 | 5.413 | 8.754 | 14.325 |
| | CAPM beta | 5.723 | 7.648 | 10.569 | 14.355 | 19.127 |
| | Jensen's alpha | -0.042 | -0.075 | -0.137 | -0.229 | -0.361 |
| $\omega = 0.12$ | Equity premium | 0.328 | 0.481 | 0.710 | 1.014 | 1.404 |
| | Equity vol | 2.375 | 3.753 | 6.436 | 11.464 | 20.895 |
| | CAPM beta | 6.097 | 8.649 | 12.955 | 19.382 | 28.764 |
| | Jensen's alpha | 0.042 | 0.076 | 0.102 | 0.104 | 0.054 |

Panel A: Gamma process



Panel B: Inverse Gaussian process

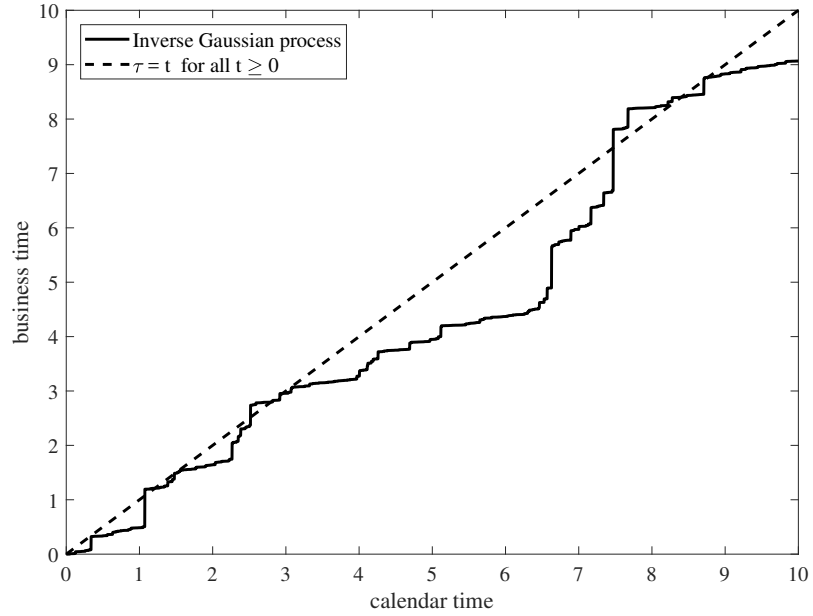
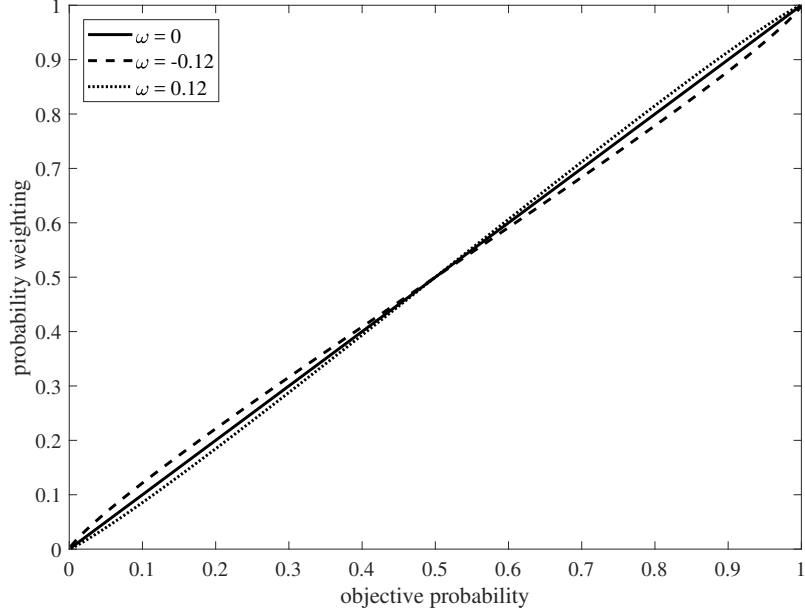


Figure 1: Sample Paths of Business Time

Panels A and B plot sample paths of the gamma process and the inverse Gaussian process with $\kappa = 0.3970$ as business time, respectively. In the panels, the solid line shows each stochastic process, while the dashed line shows the case when business time always equals calendar time.

Panel A: Probability weighting in VG



Panel B: Probability weight in NIG

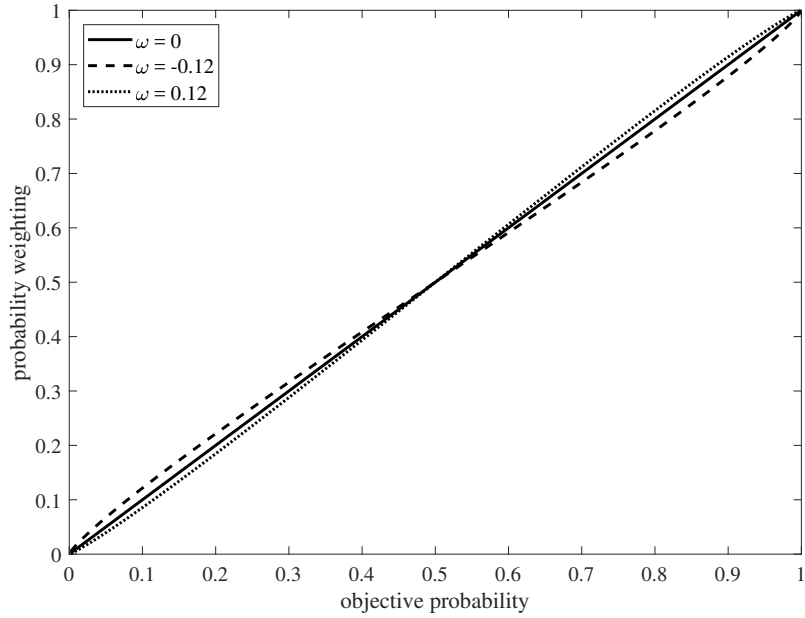
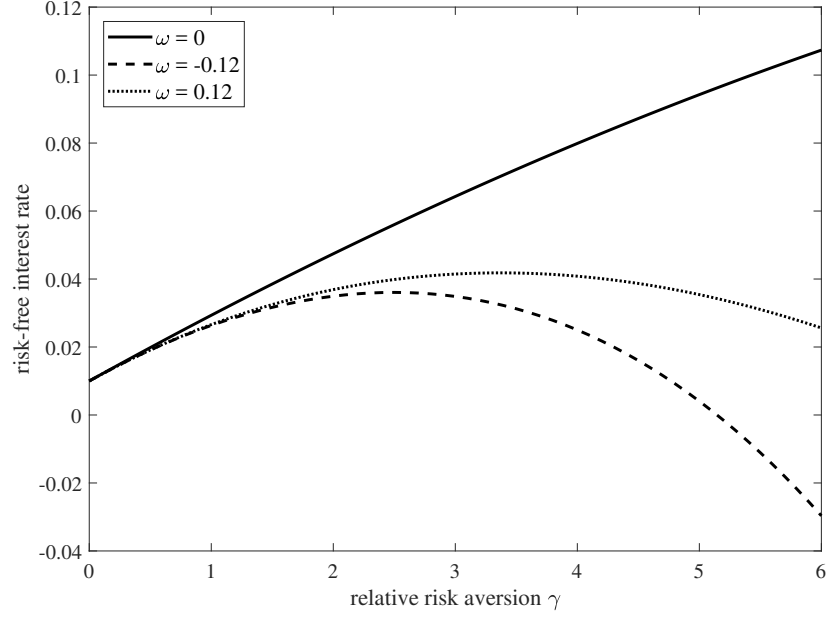


Figure 2: Probability Weight Functions

Panels A and B plot probability weighting functions for VG and NIG, respectively. In the panels, the solid line shows probability weighting for the non-biased investor ($\omega = 0$), the dashed line for the pessimistic investor ($\omega = -0.12$), and the dotted line for the optimistic investor ($\omega = 0.12$).

Panel A: Risk-free interest rate in VG



Panel B: Risk-free interest rate in NIG

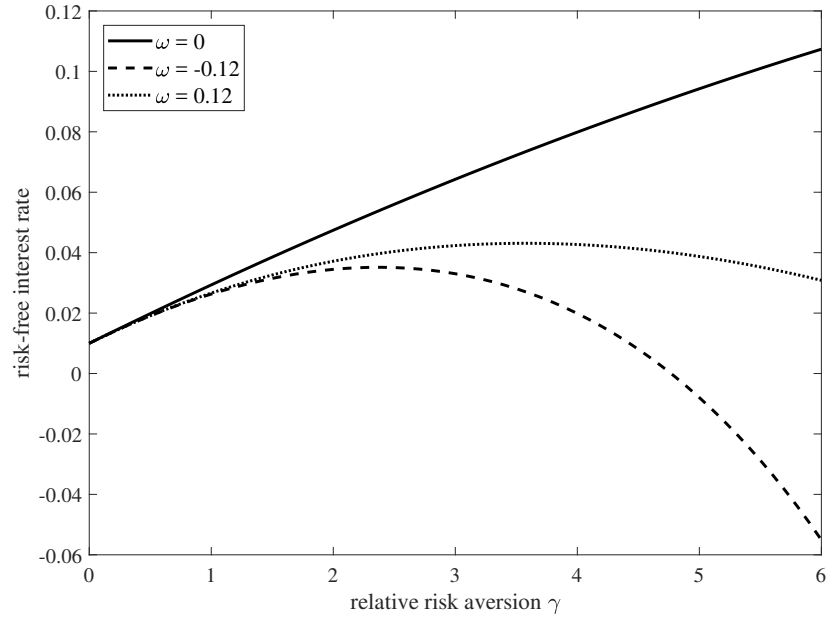
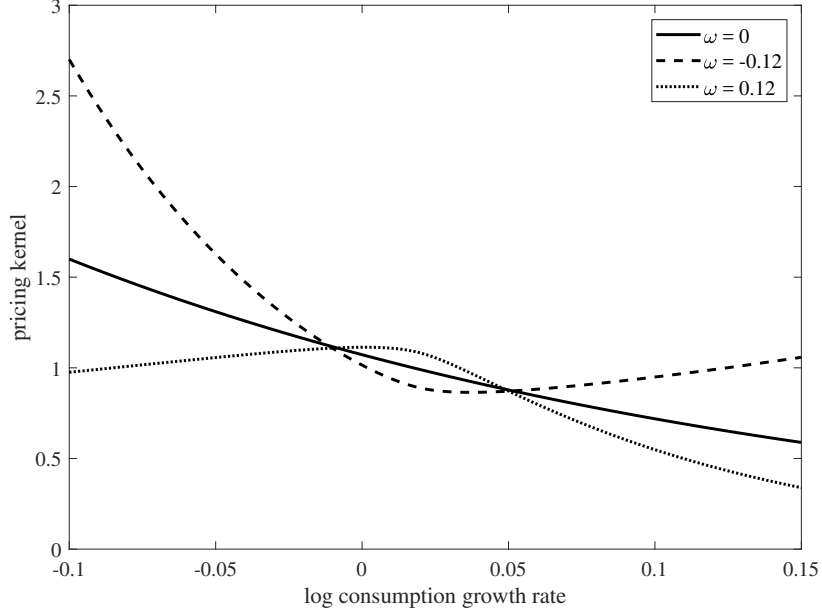


Figure 3: Risk-free Interest Rates

Panels A and B plot risk-free interest rates across relative risk aversion γ under VG and NIG, respectively. In the panels, the solid line shows interest rate for the non-biased investor ($\omega = 0$), the dashed line for the pessimistic investor ($\omega = -0.12$), and the dotted line for the optimistic investor ($\omega = 0.12$).

Panel A: Pricing kernel in VG



Panel B: Pricing kernel in NIG

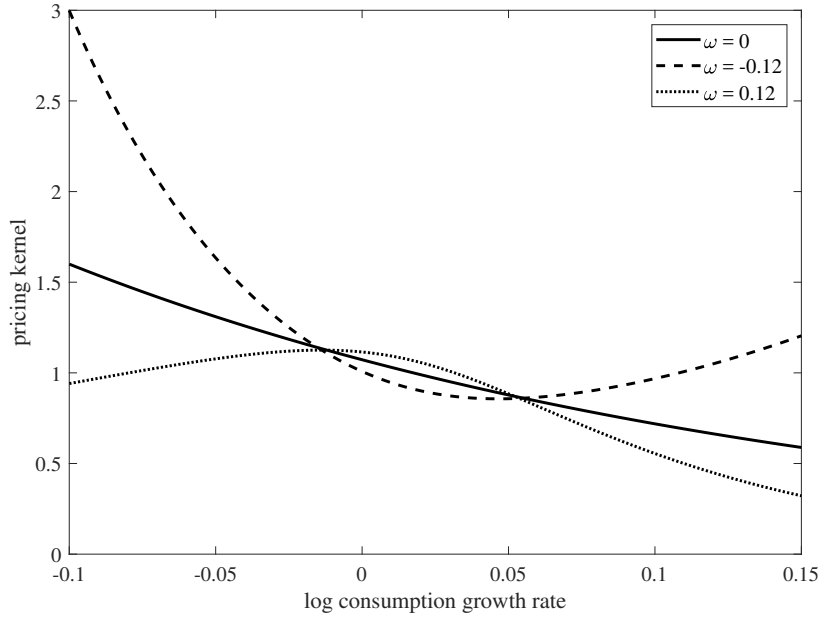


Figure 4: Projections of Pricing Kernels onto Consumption Growth Rate

Panels A and B plot the projections of pricing kernels onto the log consumption growth rate for VG and NIG, respectively. In the panels, the solid line shows pricing kernel for the non-biased investor ($\omega = 0$), the dashed line for the pessimistic investor ($\omega = -0.12$), and the dotted line for the optimistic investor ($\omega = 0.12$).

4 Conclusion

This paper suggests incorporating investor probability weighting and firm default risk into a consumption-based asset pricing model. The model offers a unified solution, with a number of seemingly anomalous patterns observed on financial markets.

As shown in the analysis, mild overweighting of probability on tail events has the potential to resolve the risk-free rate puzzle, while the financial leverage effect on equity values could reconcile both the equity premium and excess volatility puzzles. Furthermore, the analysis sheds light on anomalous patterns observed on financially distressed stocks. The simulations demonstrate the combination of mild probability weighting and higher default risk causes a cross-sectional relationship that expected excess returns on distressed stocks are decreasing in deterioration of the default risk. Strict nonlinearity of distressed stock returns against consumption growth rates create the anomalous correlation structure among them. The model also predicts large CAPM betas and negative Jensen's alphas for the distressed stocks. The implications derived from the model can help us better understand financial markets.

Finally, I acknowledge there are a number of directions for future research. A natural direction would be to extend the framework in this paper to a time-inhomogeneous model. This implies that random shocks in the economy are no longer independent and identically distributed. Although the analysis based on the extended model will be technically more difficult, it might predict the term structure of interest rates. Moreover, it might be challenging to expand the investigation object into options markets. In the framework of this paper, the options contract written on a stock can be thought of as a *compound* option written on the aggregate consumption, that is, the option of an option. Another interesting direction is further empirical analyses, such as testing the model for economic disasters such as the subprime loan crisis. Numerous firms suffer from financial distress during financial crises. Furthermore, the model is desirable to be used for the empirical analysis of financial markets other than the U.S. market.

A Properties of Lévy Processes

A stochastic process $X := (X_t)_{t \geq 0}$ on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ with values in \mathbf{R} such that $X_0 = 0$ is called a *Lévy process* if it possesses the following properties: (i) X is adapted to the filtration $(\mathcal{F}_t)_{t \geq 0}$. (ii) The sample paths of X are right continuous and left limits. (iii) For $0 \leq t < u$, $X_u - X_t$ is independent of \mathcal{F}_t and has the same distribution as X_{u-t} .

A.1 Lévy-Khintchine Formula

Lemma A.1 (Lévy-Khintchine Formula) Let X be a Lévy process with values in \mathbf{R} . The characteristic function of the distribution of X_t has the form⁴

$$\Phi_{X_t}(\theta) := \mathbb{E} [e^{i\theta X_t}] = e^{t\varphi_X(\theta)}, \quad (\text{A.1})$$

where $\theta \in \mathbf{R}$ and the function φ_X called the characteristic exponent is given by

$$\varphi_X(\theta) = i\alpha\theta - \frac{1}{2}\beta\theta^2 + \int_{-\infty}^{\infty} (e^{i\theta x} - 1 - i\theta x \mathbf{1}_{\{|x| \leq 1\}}) \Pi(dx), \quad \text{for } \theta \in \mathbf{R}. \quad (\text{A.2})$$

⁴In Appendices A, C, and D, I omit superscripts on functions and operators denoting probability measures, because these appendices provides mathematically general discussions. The results derived below are held under suitable probability measures.

Here, $\alpha \in \mathbf{R}$ and $\beta \geq 0$ are constants, and Π is a positive Radon measure on $\mathbf{R} \setminus \{0\}$ satisfying

$$\int_{-\infty}^{\infty} (1 \wedge x^2) \Pi(dx) < \infty.$$

Parameter β is called the *Gaussian coefficient* and the measure Π is known as the *Lévy measure*. The Gaussian coefficient β is the constant variance of the continuous component of the Lévy process and the Lévy measure Π determines its jump structure. The proof of the Lévy-Khintchine formula can be found, for example, on pages 37-45 of Sato (1999). The *Laplace exponent* of X is defined as $\mathcal{L}_X(\theta) := \varphi_X(-i\theta)$, which is equivalent to the cumulant generating function of the distribution of X_1 .

The business time $\tau := (\tau_t)_{t \geq 0}$ is defined as an increasing Lévy process that satisfies $\alpha \geq 0$, $\beta = 0$, and $\Pi((-\infty, 0]) = 0$. Thence, the business time has no diffusion component, only positive jumps with finite variation, and a positive drift.

A.2 Business Time

Let τ be business time. The moment generating function of τ_t is given by

$$\Psi_{\tau_t}(\theta) := \mathbb{E} [e^{\theta \tau_t}] = e^{t \mathcal{L}(\theta)},$$

for all $\theta \in (-\infty, a]$, where $a \geq 0$ is some constant such that the Laplace exponent $\mathcal{L}(\theta)$ is well defined and

$$\mathcal{L}(\theta) = \alpha\theta + \int_0^{\infty} (e^{\theta x} - 1) \Pi(dx).$$

Table A.1 exhibits the Lévy measures and the Laplace exponents of two well known increasing Lévy processes to model the business time; gamma process and inverse Gaussian process. They have only one parameter $\kappa > 0$ with $\alpha = 0$ and $\mathbb{E} [\tau_t] = t$ for every $t \geq 0$.

Table A.1: Examples of Business Time

The table exhibits the Lévy measures and the Laplace exponents of gamma process and inverse Gaussian process, both of which have parameter $\kappa > 0$. These processes are typical examples for modeling business time.

| Business time | Lévy measure $\Pi(dx)$ | Laplace exponent $\mathcal{L}(\theta)$ |
|--------------------------|--|--|
| Gamma process | $\frac{\kappa}{x} e^{-\kappa x} \mathbf{1}_{\{x>0\}} dx$ | $-\frac{1}{\kappa} \log(1 - \kappa\theta)$ |
| Inverse Gaussian process | $\frac{\kappa}{x^{3/2}} e^{-\kappa x} \mathbf{1}_{\{x>0\}} dx$ | $-\frac{1}{\kappa} (\sqrt{1 - 2\kappa\theta} - 1)$ |

Lemma A.2 (Properties of Laplace Exponent) Let $\mathcal{L}(\theta)$ be the Laplace exponent of business time τ . Suppose that $\mathcal{L}(\theta)$ is twice differentiable on $(-\infty, a]$. Then, $\mathcal{L}(\theta)$ is an increasing and convex function on $(-\infty, a]$ and $\mathcal{L}(0) = 0$. That is, $\mathcal{L}(\theta)$ is non-positive on $(-\infty, 0)$, but non-negative on $(0, a]$.

Proof of Lemma A.2: Because τ is a non-negative process, it satisfies, for any $t \geq 0$

$$\frac{d}{d\theta} \mathbb{E} [e^{\theta \tau_t}] = \mathbb{E} [\tau_t e^{\theta \tau_t}] \geq 0.$$

Hence, it must be held that, for any $t \geq 0$,

$$\frac{d}{d\theta} e^{t\mathcal{L}(\theta)} = t\mathcal{L}'(\theta) e^{t\mathcal{L}(\theta)} \geq 0.$$

Therefore, $\mathcal{L}'(\theta) \geq 0$. Next, for any $t \geq 0$, one has

$$\frac{d^2}{d\theta^2} \mathbb{E} [e^{\theta\tau_t}] = \mathbb{E} [\tau_t^2 e^{\theta\tau_t}] \geq 0.$$

Consequently, it must satisfy that, for any $t \geq 0$,

$$\frac{d^2}{d\theta^2} e^{t\mathcal{L}(\theta)} = (t\mathcal{L}''(\theta) + t^2\mathcal{L}'(\theta)^2) e^{t\mathcal{L}(\theta)} \geq 0. \quad (\text{A.3})$$

Here, $f(t) := t\mathcal{L}''(\theta) + t^2\mathcal{L}'(\theta)^2$. The inequality (A.3) indicates $f(t) \geq 0$ for all $t \geq 0$. If $\mathcal{L}''(\theta) < 0$, then $f(t) < 0$ for $0 < t < -\mathcal{L}''(\theta)\mathcal{L}'(\theta)^{-2}$. This is a contradiction. On the other hand, if $\mathcal{L}''(\theta) \geq 0$, then $f(t) < 0$ for every $t \geq 0$. Finally, $\mathcal{L}(0) = 0$ is trivial. \square

Lemma A.3 (Subordination) Let $Y := (Y(t))_{t \geq 0}$ be a drifted Brownian motion, that is, $Y(t) = \omega t + \sigma W(t)$ for every $t \geq 0$, where $\omega \in \mathbf{R}$ and $\sigma > 0$ are constants and $W := (W(t))_{t \geq 0}$ is a one-dimensional standard Brownian motion. Let τ be business time with Laplace exponent $\mathcal{L}(\theta)$. Then, the stochastic process $X := (X_t)_{t \geq 0}$ defined by $X_t = Y(\tau_t) = \omega\tau_t + \sigma W(\tau_t)$ for every $t \geq 0$ is a Lévy process and its characteristic exponent has the form

$$\varphi_X(\theta) = \mathcal{L}(-\sigma^2\theta^2/2 + i\omega\theta).$$

The Lévy process X is thought of as a time-changed Brownian motion by business time τ . This transformation is called the *subordination*. The proof of Lemma A.3 can be found, for example, on pages 108-109 of Cont & Tankov (2004).

A.3 Esscher Transform

Let $X = (X_t)_{t \geq 0}$ be a Lévy process. The Esscher transform by X from the original measure \mathbb{P} to a new measure \mathbb{Q} with parameter $a \in \mathbf{R}$ is defined by the Radon-Nikodym derivative

$$\left. \frac{d\mathbb{Q}}{d\mathbb{P}} \right|_t := \frac{e^{aX_t}}{\mathbb{E}^\mathbb{P}[e^{aX_t}]}.$$

By the Esscher transform, the Lévy-Khintchine representation (A.2) is transformed into

$$\begin{aligned} \varphi_X^\mathbb{Q}(\theta) &= \varphi_X^\mathbb{P}(\theta - ia) - \psi_X^\mathbb{P}(a) \\ &= i\tilde{\alpha}\theta - \frac{1}{2}\beta\theta^2 + \int_{-\infty}^{\infty} (e^{i\theta x} - 1 - i\theta x \mathbf{1}_{\{|x| \leq 1\}}) \tilde{\Pi}(dx), \end{aligned}$$

where

$$\tilde{\alpha} := \alpha + a\beta + \int_{-1}^1 x(e^{ax} - 1) \Pi(dx), \quad \text{and} \quad \tilde{\Pi}(x) := e^{ax} \Pi(x).$$

The transformed Lévy measure $\tilde{\Pi}$ is exponentially tilted, which causes to change the jump structure of the Lévy process X under the new probability measure \mathbb{Q} , while the Gaussian coefficient β is unvaried. When $a < 0$, transforming into the Lévy measure $\tilde{\Pi}$ is called the *tempering* with parameter a and X under \mathbb{Q} is known as the *tempered Lévy process*.

Table A.2: Characteristic Exponents and Cumulants of Consumption Growth Rate Process

The table exhibits the Laplace exponents $\varphi_R^*(\theta)$ of the log consumption growth rate process modeled by Brownian motion (BM), variance gamma process (VG), and normal inverse Gaussian process (NIG), in which the superscript * takes the probability measure \mathbb{P} , \mathbb{P}^S , or \mathbb{Q} . Cumulants of R_1 are also shown here. They are plugged into suitable model parameters σ_* , κ_* , and ω_* corresponding to each probability measure listed in Table A.3.

| | BM | VG | NIG |
|-----------------------|---|---|--|
| $\varphi_R^*(\theta)$ | $i(\mu + \omega_*)\theta - \frac{1}{2}\sigma_*^2\theta^2$ | $i\mu\theta - \frac{1}{\kappa_*}\log\left(1 + \frac{1}{2}\kappa_*\sigma_*^2\theta^2 - i\omega_*\kappa_*\theta\right)$ | $i\mu\theta + \frac{1}{\kappa_*} - \frac{1}{\kappa_*}\sqrt{1 + \kappa_*\sigma_*^2\theta^2} - 2i\omega_*\kappa_*\theta$ |
| 1st cumulant | $\mu + \omega_*$ | $\mu + \omega_*$ | $\mu + \omega_*$ |
| 2nd cumulant | σ_*^2 | $\sigma_*^2 + \omega_*^2\kappa_*$ | $\sigma_*^2 + \omega_*^2\kappa_*$ |
| 3rd cumulant | 0 | $3\sigma_*^2\omega_*\kappa_* + 2\omega_*^3\kappa_*^2$ | $3\sigma_*^2\omega_*\kappa_* + 3\omega_*^3\kappa_*^2$ |
| 4th cumulant | 0 | $3\sigma_*^4\kappa_* + 6\omega_*^4\kappa_*^3 + 12\sigma_*^2\omega_*^2\kappa_*^2$ | $3\sigma_*^4\kappa_* + 15\omega_*^4\kappa_*^3 + 18\sigma_*^2\omega_*^2\kappa_*^2$ |

B Proofs

B.1 Proof of Lemma 1

From the definition of the risk-neutral probability \mathbb{Q} and (A.1), one has

$$\begin{aligned}\Phi_{R_t}^{\mathbb{Q}}(\theta) &= \mathbb{E}^{\mathbb{Q}}[e^{i\theta R_t}] = e^{rt} \mathbb{E}^{\mathbb{P}}[\mathcal{M}_t^S e^{i\theta R_t}] \\ &= \exp\{i\theta\mu t - \mathcal{L}(\gamma^2\sigma^2/2 - \gamma\omega)t\} \mathbb{E}^{\mathbb{P}}[\exp\{(i\theta - \gamma)\sigma W(\tau_t) - \gamma\omega\tau_t\}] \\ &= e^{t\varphi_R^{\mathbb{Q}}(\theta)}.\end{aligned}$$

Proposition 1 and Lemma A.2 are applied to the third and the last equalities of the above equation, respectively. In a similar manner, one can obtain the representation (8) that offers the characteristic exponent of R_t under the subjective probability. \square

B.2 Proof of Proposition 1

Following to the Euler equation, the price of a zero-coupon risk-free bond can be represented as

$$\begin{aligned}B(t) &= \mathbb{E}^{\mathbb{P}}[\mathcal{M}_t^S] = \exp\{-(\delta + \gamma\mu - \gamma\omega)t\} \mathbb{E}^{\mathbb{P}}[\exp\{-\gamma\sigma W(\tau_t) - \gamma\omega\tau_t\}] \\ &= \exp\{-(\delta + \gamma\mu - \gamma\omega)t\} \exp\{\mathcal{L}(\gamma^2\sigma^2/2 - \gamma\omega)t\}.\end{aligned}$$

Lemma A.2 is used in the last equality of the above equation. \square

B.3 Proof of Proposition 2

Using the relation (A.1), one has

$$\mathbb{E}_t^{\mathbb{P}}\left[\frac{\mathcal{M}_u^S}{\mathcal{M}_t^S} Z_u\right] = Z_t e^{-\delta(u-t)} \mathbb{E}_t^{\mathbb{P}}\left[\left(\frac{C_u^S}{C_t^S}\right)^{-\gamma} \left(\frac{Z_u}{Z_t}\right)\right] = Z_t e^{-\xi(u-t)}.$$

Because of $\xi > 0$ by Assumption 1, the firm value (12) holds

$$V_t = \int_t^\infty Z_t e^{-\xi(u-t)} du = \frac{Z_t}{\xi}.$$

\square

C Some Lemmas

This appendix provides some lemmas used for computation in the simulation and calibration. Let $\mathfrak{F}^{-1}[f](k) := \frac{1}{2\pi} \int_{-\infty}^\infty e^{-i\theta k} f(\theta) d\theta$ be the inverse Fourier transform of a function f with respect to parameter θ , where f is a complex-valued function of parameter θ .

Lemma C.1 For any $k, a, b \in \mathbf{R}$ and $T \geq 0$, define

$$A_{a,b}(k, T) := \mathbb{E}[e^{aR_T + bk} \mathbf{1}_{\{R_T > k\}}].$$

Then, it satisfies

$$A_{a,b}(k, T) = \mathfrak{F}^{-1}[f_{a,b}](k) + e^{T\psi_R(a) + bk} \mathbf{1}_{\{q > k\}}, \quad (\text{C.1})$$

Table A.3: Model Parameters of Consumption Growth Rate Process

Panel A exhibits the original parameters for the consumption growth rate process under the objective probability measure \mathbb{P} molded by Brownian motion (BM), variance gamma process (VG), and normal inverse Gaussian process (NIG). Panels B and C show the transformed parameters under the subjective probability measure \mathbb{P}^S and the risk-neutral probability measure \mathbb{Q} , respectively.

| Panel A: Objective parameters | | | |
|-------------------------------|----------|----------|----------|
| | BM | VG | NIG |
| $\sigma_{\mathbb{P}}$ | σ | σ | σ |
| $\kappa_{\mathbb{P}}$ | | κ | κ |
| $\omega_{\mathbb{P}}$ | 0 | 0 | 0 |

| Panel B: Subjective parameters | | | |
|--------------------------------|----------|--|--|
| | BM | VG | NIG |
| $\sigma_{\mathcal{S}}$ | σ | $\frac{\sigma}{\sqrt{1 + \omega\kappa\gamma}}$ | $\frac{\sigma}{\sqrt[4]{1 + 2\omega\kappa\gamma}}$ |
| $\kappa_{\mathcal{S}}$ | | κ | $\frac{\kappa}{\sqrt{1 + 2\omega\kappa\gamma}}$ |
| $\omega_{\mathcal{S}}$ | 0 | 0 | 0 |

| Panel C: Risk-neutral parameters | | | |
|----------------------------------|-------------------|---|---|
| | BM | VG | NIG |
| $\sigma_{\mathbb{Q}}$ | σ | $\frac{\sigma}{\sqrt{1 - \kappa\sigma^2\gamma^2/2 + \omega\kappa\gamma}}$ | $\frac{\sigma}{\sqrt[4]{1 - \kappa\sigma^2\gamma^2 + 2\omega\kappa\gamma}}$ |
| $\kappa_{\mathbb{Q}}$ | | κ | $\frac{\kappa}{\sqrt{1 - \kappa\sigma^2\gamma^2 + 2\omega\kappa\gamma}}$ |
| $\omega_{\mathbb{Q}}$ | $-\sigma^2\gamma$ | $\frac{-\sigma^2\gamma}{1 - \kappa\sigma^2\gamma^2/2 + \omega\kappa\gamma}$ | $\frac{-\sigma^2\gamma}{\sqrt{1 - \kappa\sigma^2\gamma^2 + 2\omega\kappa\gamma}}$ |

where $q \in \mathbf{R}$ is an arbitrary control parameter⁵ and

$$f_{a,b}(\theta) := \frac{\Phi_{R_T}(\theta - i[a+b]) - e^{(i\theta+b)q}\Psi_{R_T}(a)}{i\theta + b}.$$

Proof of Lemma C.1: Define the function h as

$$h(k) := \mathbb{E} \left[e^{aR_T + bk} \mathbf{1}_{\{R_T > k\}} \right] - e^{T\psi_R(a) + bk} \mathbf{1}_{\{q > k\}}.$$

Because $\mathbb{E} \left[e^{aR_T} \right] = e^{T\psi_R(a)}$, one has

$$h(k) = \mathbb{E} \left[e^{aR_T + bk} \left(\mathbf{1}_{\{R_T > k\}} - \mathbf{1}_{\{q > k\}} \right) \right].$$

Denote the Fourier transform of h with respect to parameter k by $f_{a,b}(\theta) := \mathfrak{F}[h](\theta)$. Then, it satisfies

$$\begin{aligned} f_{a,b}(\theta) &= \int_{-\infty}^{\infty} e^{i\theta k} h(k) dk = \mathbb{E} \left[\int_q^{R_T} e^{i\theta k} e^{aR_T + bk} dk \right] \\ &= \frac{1}{i\theta + b} \mathbb{E} \left[e^{(i\theta + a + b)R_T} - e^{aR_T + (i\theta + b)q} \right] \\ &= \frac{\Phi_{R_T}(\theta - i[a+b]) - e^{(i\theta + b)q}\Psi_{R_T}(a)}{i\theta + b}. \end{aligned}$$

By the inverse Fourier transform of $f_{a,b}$, the formula (C.1) is obtained. \square

Lemma C.2 For any $a, r \in \mathbf{R}$ and $T \geq 0$, define

$$I_a(r, T) := \mathbb{E} \left[\int_0^T e^{-ru} e^{aR_u} du \right].$$

Then, it satisfies

$$I_a(r, T) = \frac{1 - e^{-rT}\Psi_{R_T}(a)}{r - \psi_R(a)}.$$

Proof of Lemma C.2: It is trivial. \square

Lemma C.3 For any $a, b, r \in \mathbf{R}$ and $T \geq 0$, define

$$J_{a,b}(r_1, r_2, T) := \mathbb{E} \left[\left(\int_0^T e^{-r_1 u} e^{aR_u} du \right) \left(\int_0^T e^{-r_2 u} e^{bR_u} du \right) \right].$$

Then, it satisfies

$$\begin{aligned} J_{a,b}(r_1, r_2, T) &= \frac{1}{r_2 + \psi_R(a) - \psi_R(a+b)} \left[I_a(r_1, T) - \frac{1 - e^{-(r_1+r_2)T}\Psi_{R_T}(a+b)}{r_1 + r_2 - \psi_R(a+b)} \right] \\ &\quad + \frac{1}{r_1 + \psi_R(b) - \psi_R(a+b)} \left[I_b(r_2, T) - \frac{1 - e^{-(r_1+r_2)T}\Psi_{R_T}(a+b)}{r_1 + r_2 - \psi_R(a+b)} \right]. \end{aligned}$$

⁵A standard choice for control parameter q is $T\psi_R(1)$, which is the mean value of R_T . See the chapter 11.1.3 in Cont & Tankov (2004) and the appendix A in Yamazaki (2018) for example.

Proof of Lemma C.3: Define the function $h_{a,b}$ as

$$h_{a,b}(u, s) := \mathbb{E} \left[e^{aR_u + bR_s} \right].$$

for any $0 \leq s < u \leq T$. Then, by the law of iterated expectations and the independent and stationary properties of increments of Lévy processes, one has

$$\begin{aligned} h_{a,b}(u, s) &= \mathbb{E} \left[e^{(a+b)R_s} \mathbb{E}_s \left[e^{a(R_u - R_s)} \right] \right] = \mathbb{E} \left[e^{(a+b)R_s} e^{(u-s)\psi_R(a)} \right] \\ &= e^{u\psi_R(a) - s(\psi_R(a) - \psi_R(a+b))}. \end{aligned}$$

Since

$$\begin{aligned} J_{a,b}(r_1, r_2, T) &= \mathbb{E} \left[\int_0^T \int_0^u e^{-r_1 u} e^{aR_u} e^{-r_2 s} e^{bR_s} ds du + \int_0^T \int_0^u e^{-r_2 u} e^{bR_u} e^{-r_1 s} e^{aR_s} ds du \right] \\ &= \int_0^T \int_0^u e^{-(r_1 u + r_2 s)} h_{a,b}(u, s) ds du + \int_0^T \int_0^u e^{-(r_2 u + r_1 s)} h_{b,a}(u, s) ds du, \end{aligned}$$

the proof is completed. \square

Lemma C.4 For any $a, b, r, k \in \mathbf{R}$ and $T \geq 0$, define

$$G_{a,b}(r, k, T) = \mathbb{E} \left[\left(\int_0^T e^{-ru} e^{aR_u} du \right) e^{bR_T} \mathbf{1}_{\{R_T > k\}} \right].$$

Then, it satisfies

$$G_{a,b}(r, k, T) = \mathfrak{F}^{-1} [g_{a,b}] (k) + \frac{\Psi_{R_T}(b) - e^{-rT} \Psi_{R_T}(a+b)}{r + \psi_R(b) - \psi_R(a+b)} \mathbf{1}_{\{q > k\}}, \quad (\text{C.2})$$

where $q \in \mathbf{R}$ is an arbitrary control parameter and

$$g_{a,b}(\theta) = \frac{1}{i\theta} \left[\frac{\Phi_{R_T}(\theta - ib) - e^{-rT} \Phi_{R_T}(\theta - i[a+b])}{r + \varphi_R(\theta - ib) - \varphi_R(\theta - i[a+b])} - e^{i\theta q} \frac{\Psi_{R_T}(b) - e^{-rT} \Psi_{R_T}(a+b)}{r + \psi_R(b) - \psi_R(a+b)} \right].$$

Proof of Lemma C.4: First of all, define the function h_1 as

$$h_1(k, u) := \mathbb{E}_u \left[e^{b(R_T - R_u)} \mathbf{1}_{\{R_T > k\}} \right] - e^{(T-u)\psi_R(b)} \mathbf{1}_{\{q > k\}}.$$

Because $\mathbb{E}_u [e^{b(R_T - R_u)}] = e^{(T-u)\psi_R(b)}$, one has

$$h_1(k, u) = \mathbb{E}_u \left[e^{b(R_T - R_u)} (\mathbf{1}_{\{R_T > k\}} - \mathbf{1}_{\{q > k\}}) \right].$$

Denote the Fourier transform of h_1 with respect to parameter k by $\hat{h}_1(\theta, u) := \mathfrak{F}[h_1](\theta)$. Then, it satisfies

$$\begin{aligned} \hat{h}_1(\theta, u) &= \int_{-\infty}^{\infty} e^{i\theta k} h_1(k, u) dk = \mathbb{E}_u \left[\int_q^{R_T} e^{i\theta k} e^{b(R_T - R_u)} dk \right] \\ &= \frac{1}{i\theta} \mathbb{E}_u \left[e^{i\theta R_u + (i\theta + b)(R_T - R_u)} - e^{i\theta q + b(R_T - R_u)} \right] \\ &= \frac{1}{i\theta} \left[e^{i\theta R_u + (T-u)\varphi_R(\theta - ib)} - e^{i\theta q + (T-u)\psi_R(b)} \right]. \end{aligned}$$

Therefore, by the inverse Fourier transform of \hat{h}_1 with respect to parameter θ , one has

$$\mathbb{E}_u \left[e^{b(R_T - R_u)} \mathbf{1}_{\{R_T > k\}} \right] = \mathfrak{F}^{-1} \left[\hat{h}_1 \right] (k) + e^{(T-u)\psi_R(b)} \mathbf{1}_{\{q > k\}}. \quad (\text{C.3})$$

Next, define the function h_2 as

$$h_2(k, u) := \mathbb{E} \left[e^{aR_u} e^{bR_T} \mathbf{1}_{\{R_T > k\}} \right].$$

Using the law of iterated expectations and (C.3), it satisfies

$$\begin{aligned} h_2(k, u) &= \mathbb{E} \left[e^{(a+b)R_u} \mathbb{E}_u \left[e^{b(R_T - R_u)} \mathbf{1}_{\{R_T > k\}} \right] \right] \\ &= \mathbb{E} \left[e^{(a+b)R_u} \mathfrak{F}^{-1} \left[\hat{h}_1 \right] (k) \right] + e^{(T-u)\psi_R(b)} \mathbf{1}_{\{q > k\}} \mathbb{E} \left[e^{(a+b)R_u} \right] \\ &= \mathfrak{F}^{-1} \left[\hat{h}_2 \right] (k) + e^{T\psi_R(b)} e^{-u(\psi_R(b) - \psi_R(a+b))} \mathbf{1}_{\{q > k\}}, \end{aligned}$$

where $\hat{h}_2(\theta, u) := \mathbb{E} \left[e^{(a+b)R_u} \hat{h}_1(\theta, u) \right]$. Plugging into $\hat{h}_1(\theta, u)$, one has

$$\begin{aligned} \hat{h}_2(\theta, u) &= \frac{1}{i\theta} \left[e^{(T-u)\varphi_R(\theta - ib)} \mathbb{E} \left[e^{i\theta(a+b)R_u} \right] - e^{i\theta q + (T-u)\psi_R(b)} \mathbb{E} \left[e^{(a+b)R_u} \right] \right] \\ &= \frac{1}{i\theta} \left[e^{T\varphi_R(\theta - ib)} e^{-u(\varphi_R(\theta - ib) - \varphi_R(\theta - i[a+b]))} - e^{i\theta q + T\psi_R(b)} e^{-u(\psi_R(b) - \psi_R(a+b))} \right]. \end{aligned}$$

Finally, one has

$$\begin{aligned} \mathbb{E} \left[\left(\int_0^T e^{-ru} e^{R_u} du \right) e^{R_T} \mathbf{1}_{\{R_T > k\}} \right] &= \int_0^T e^{-ru} h_2(k, u) du \\ &= \mathfrak{F}^{-1} \left[\int_0^T e^{-ru} \hat{h}_2(\theta, u) du \right] (k) + e^{T\psi_R(b)} \mathbf{1}_{\{q > k\}} \int_0^T e^{-u(r + \psi_R(b) - \psi_R(a+b))} du \\ &= \mathfrak{F}^{-1} [g_{a,b}] (k) + \frac{\Psi_{R_T}(b) - e^{-rT} \Psi_{R_T}(a+b)}{r + \psi_R(b) - \psi_R(a+b)} \mathbf{1}_{\{q > k\}}, \end{aligned}$$

where

$$\begin{aligned} g_{a,b}(\theta) &:= \int_0^T e^{-ru} \hat{h}_2(\theta, u) du \\ &= \frac{1}{i\theta} \left[\frac{\Phi_{R_T}(\theta - ib) - e^{-rT} \Phi_{R_T}(\theta - i[a+b])}{r + \varphi_R(\theta - ib) - \varphi_R(\theta - i[a+b])} - e^{i\theta q} \frac{\Psi_{R_T}(b) - e^{-rT} \Psi_{R_T}(a+b)}{r + \psi_R(b) - \psi_R(a+b)} \right]. \end{aligned}$$

By the inverse Fourier transform of g_1 , the formula (C.2) is obtained. \square

Lemma C.5 For any $a, b, r, k \in \mathbf{R}$ and $T \geq 0$, define

$$H_{a,b}(r, k, T) := \mathbb{E} \left[\left(\int_0^T e^{-ru} e^{aR_u} du \right) e^{bk} \mathbf{1}_{\{R_T > k\}} \right].$$

Then, it satisfies

$$H_{a,b}(r, k, T) = \mathfrak{F}^{-1} [h_{a,b}] (k) + \frac{1 - e^{-rT} \Psi_{R_T}(a)}{r - \psi_R(1)} e^{bk} \mathbf{1}_{\{q > k\}}, \quad (\text{C.4})$$

where $q \in \mathbf{R}$ is an arbitrary control parameter and

$$h_{a,b}(\theta) := \frac{1}{i\theta + b} \left[\frac{\Phi_{R_T}(\theta - ib) - e^{-rT} \Phi_{R_T}(\theta - i[a+b])}{r + \varphi_R(\theta - ib) - \varphi_R(\theta - i[a+b])} - e^{i\theta q + bT\psi_R(1)} \frac{1 - e^{-rT} \Psi_{R_T}(a)}{r - \psi_R(a)} \right].$$

Proof of Lemma C.5: One can obtain the representation (C.4) in a similar manner to the proof of Lemma C.4. \square

D Moments of Λ_T

The computation formulas for moments of Λ_T defined in (19) are needed to obtain fundamental statistics for stock returns in Section 2.8. However, instead of Λ_T , consider moments of $\Upsilon_T := \Lambda_T / (V_0 e^{rT})$ for simplicity of notations. The first and the second moments of Υ_T are given by

$$\mathbb{E}[\Upsilon_T] = \xi \eta I_\nu(\bar{r}, T) + e^{-\bar{r}T} [A_{\nu,0}(k, T) - A_{0,\nu}(k, T)],$$

and

$$\begin{aligned} \mathbb{E}[(\Upsilon_T)^2] &= \xi^2 \eta^2 J_{\nu,\nu}(\bar{r}, \bar{r}, T) + 2\xi \eta e^{-\bar{r}T} [G_{\nu,\nu}(\bar{r}, k, T) - H_{\nu,\nu}(\bar{r}, k, T)] \\ &\quad + e^{-2\bar{r}T} [A_{2\nu,0}(k, T) - 2A_{\nu,\nu}(k, T) + A_{0,2\nu}(k, T)], \end{aligned}$$

respectively, where $\bar{r} := r + \mu\nu - \zeta$ and $k := \nu^{-1} \log(F/V_0) + (\mu - \zeta\nu^{-1})T$. Next, suppose $k_1 > k_2$, where $k_l := \nu_l^{-1} \log(F_l/V_{l,0}) + (\mu - \zeta_l \nu_l^{-1})T$ for firm l . The cross moment between $\Upsilon_{1,T}$ and $\Upsilon_{2,T}$, instead of $\Lambda_{1,T}$ and $\Lambda_{2,T}$, is given by

$$\begin{aligned} \mathbb{E}[\Upsilon_{1,T} \Upsilon_{2,T}] &= \xi_1 \xi_2 \eta_1 \eta_2 J_{\nu_1, \nu_2}(\bar{r}_1, \bar{r}_2, T) \\ &\quad + \xi_1 \eta_1 e^{-\bar{r}_2 T} [G_{\nu_1, \nu_2}(\bar{r}_1, k_2, T) - H_{\nu_1, \nu_2}(\bar{r}_1, k_2, T)] \\ &\quad + \xi_2 \eta_2 e^{-\bar{r}_1 T} [G_{\nu_2, \nu_1}(\bar{r}_2, k_1, T) - H_{\nu_2, \nu_1}(\bar{r}_2, k_1, T)] \\ &\quad + e^{-(\bar{r}_1 + \bar{r}_2)T} [A_{\nu_1 + \nu_2, 0}(k_1, T) - e^{\nu_2 k_2} A_{\nu_1, 0}(k_1, T) \\ &\quad - A_{\nu_2, \nu_1}(k_1, T) + e^{\nu_2 k_2} A_{0, \nu_1}(k_1, T)], \end{aligned}$$

where $\bar{r}_l := r + \mu\nu_l - \zeta_l$ for $l = 1, 2$.

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