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(出版者 / Publisher)
法政大学比較経済研究所 / Institute of Comparative Economic Studies, Hosei University

(雑誌名 / Journal or Publication Title)
Journal of International Economic Studies

(巻 / Volume)
27

(開始ページ / Start Page)
23

(終了ページ / End Page)
45

(発行年 / Year)
2013-03

(URL)
https://doi.org/10.15002/00008720

Journal of International Economic Studies (2013), No.27, 23–45

On the Uniqueness and Stability Conditions for Two Types of Monetary Models with Recursive Utility

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Abstract

This paper explores the conditions for local dynamics and the uniqueness of a steady state in the money-in-utility-function, and transaction-costs models with recursive utility. When a money demand depends only on nominal interest rates and a monetary Brock-Gale (BG) condition holds around a steady state, the steady state is locally stable. Even in the case of the general money demand function, the local stability is sometimes possible. Furthermore, when a money demand function exists and the BG condition holds globally, the steady state is unique.

Keywords: Recursive utility, Money-in-utility-function model, Transaction-costs model, Decreasing marginal impatience

JEL Classification: O42

1. Introduction

The present paper introduces conditions for local dynamics and the uniqueness of a steady state in the money-in-utility-function (MIUF) and transaction-costs (TC) models with recursive utility in a unified manner. Several papers have analyzed the relationship between inflation and growth in the MIUF models (Sidrauski, 1967) and TC models (Saving, 1971). Most of these studies, such as those of Wang and Yip (1992) and Zhang (2000), used a utility function with a constant time preference and an endogenous labor decision and showed that higher growth rates of money lower capital in the long run.

On the other hand, Chen et al. (2008) considered a utility function with an endogenous time preference, or a recursive utility (Koopmans, 1960). Without labor-leisure trade-offs, they discovered that the MIUF (resp. TC) model predicts a positive correlation between inflation and growth (Tobin, 1965) in the case of increasing marginal impatience (IMI) with respect to real balances (resp. consumption).[1] Similarly, decreasing marginal impatience (DMI) leads to a reverse Tobin effect. Chen et al. concluded that a reverse Tobin effect was more plausible because

[1] Hayakawa (1995) showed that capital and money are invariant to monetary growth in the case of the MIUF model with recursive utility and perfect complementary of consumption and money.

there were empirical results supporting DMI in real balances (Becker and Mulligan, 1997, among others).

Chen et al. (2008) established qualitative equivalence between the MIUF and TC models in terms of ordinal and reverse Tobin effects. In order to establish such an equivalence condition, however, they conducted only a comparative static analysis in a steady state, leaving the transitional dynamics unexplored. The steady-state equilibrium might be unstable or might generate multiple equilibriums. Investigating the stability properties in the neighborhood of the steady state involves a linearized dynamic system with a Jacobian matrix. Even in the case of comparative statics, checking the sign of the characteristic roots of the matrix is required. However, Chen et al. simply assumed that the determinant is negative. The major reason for this assumption may have been that the dynamics of this system was four-dimensional and, therefore, was too complicated to analyze.

In order to avoid investigating four-dimensional dynamics directly, we use the method proposed by Obstfeld (1990) and Chang (1994). They considered a one-sector, nonmonetary, optimal growth model with recursive utility. Exploiting the facts that the value function is the same as a costate variable of the dynamic system on the optimal path and that the derivative of the value function is the other costate variable, they succeeded in reducing the dimensions of the dynamic system from three to two. Similarly, in the present paper, we reduce the dimensions of the two monetary models from four to three.

Fischer (1979) and Calvo (1979) investigated the sufficient conditions for the local stability of the MIUF model with a constant time preference. The dynamic system of the monetary model has three dimensions, with two jump variables, requiring one negative root and two positive roots in the characteristic function. In order to establish only one negative root in the dynamic system, the determinant (the products of the three roots) should be negative, and the trace (the sum of the roots) should be positive. Fischer (1979) assumed that the instantaneous utility function is concave with respect to consumption and real balances and that consumption and real balances are normal goods. Instead of the latter conditions, Calvo (1979) proposed a sufficient but simple condition, in which both goods are gross substitutes.

Since the study by Uzawa (1968), a great deal of literature has been devoted to analyzing models with recursive utility. Although Uzawa is one exception, few studies have dealt with the stability conditions of monetary models. In a different framework from the present models, he insisted that impatience should be marginally increasing in felicity.[2] Another exception is Epstein and Hynes (1983). Their MIUF model assumed that the instantaneous utility function is a negative constant, that marginal impatience is increasing and concave in consumption and real balances, and that both goods are normal. Under these assumptions, Epstein and Hynes claimed that the dynamic system is locally stable.

Even in the case of non-monetary models with recursive utility, most studies have considered only the stable case by assuming IMI in consumption or in felicity from the beginning. Das (2003) investigated the condition for stability in the case of DMI.[3] Her condition implies that local stability requires a variant of the Brock–Gale (BG) condition, in which an increase in the discount rate dominates the increase in the marginal product of capital at the steady state (Brock and Gale, 1969). Although Chen et al. (2008) assumed this BG condition, called the Correspondence Principle, in their paper, they used this condition simply to discuss the unique existence of the

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[2] He also assumed that felicity takes a positive value and that the discounting rate function is convex.

[3] See also Hirose and Ikeda (2008) for DMI.

steady state.[4] Furthermore, the BG condition that Chen et al. assumed ignored the effect of money. If a monetary variant of the BG condition is established correctly, we conjecture that this condition will play an essential role in determining a negative determinant of the matrix as well as the unique steady state.

The present paper introduces conditions for the local dynamics and the uniqueness of a steady state in the money-in-utility-function and transaction-costs models with recursive utility in a unified manner. A negative determinant of the matrix calls for a monetary variant of the BG condition. If this condition is satisfied and the money demand function is well-defined locally (resp. globally), this is an essential condition for local stability (resp. uniqueness). In addition to the BG condition, an additional condition for local stability is that money demand is a decreasing function only of the nominal interest rate, and impatience is marginally increasing in consumption and money.

When money demand is increasing in consumption, other conditions are required for each model. Both models require IMI in consumption. Since the money demand function is characterized by felicity and discounting rates in the MIUF model, local stability requires the felicity to be concave and the discounting rate to be either concave for a negative felicity or convex for a positive felicity. In the TC model, the money demand function is characterized by a transaction-costs function, and the transaction costs function should be convex for local stability.

This does not deny the possibility of DMI. The local stability with DMI is more plausible as: (i) the production function becomes more concave and the money demand function becomes less elastic in both models; (ii) the felicity taking a negative (resp. positive) value becomes more concave, and the discounting rate becomes more concave (resp. convex) in the MIUF model; and (iii) the transaction technology becomes more convex in the TC model.

The present paper provides the following three contributions. First, we establish conditions for local stability and the unique steady state, which reinforce the results of Chen et al. (2008) and are consistent with the results of other previous studies. Second, using a money demand function, we provide such conditions in the MIUF and TC models in a unified manner. Third, we demonstrate that a monetary version of the BG condition is essential in determining uniqueness and local stability.

The remainder of the present paper is organized as follows. Section 2 examines the MIUF model, and Section 3 analyzes the TC model. Each section describes the model economy and investigates the local stability around the steady state. Section 4 discusses the conditions for the uniqueness of the steady state for the MIUF and TC models, and Section 5 concludes the paper.

2. The MIUF Model

This section considers the local stability of the MIUF model with recursive utility, in which money or real balances directly consider the felicity function and the subjective discount rate function. The model structure and notations closely follow Chen et al. (2008) in order to facilitate a comparison between their results and the results of the present study. Our assumption set is slightly different from theirs because we allow the felicity function to take a positive or a negative value. The first subsection describes the model economy and the definition of the steady state. The next subsection presents the model and examines local stability. Starting with the minimal

[4] To prove the uniqueness, Chen et al. (2008) considered a special case, in which the discount rate is independent of real balances in the MIUF model. For other cases, the uniqueness is shown graphically without rigid proofs.

assumptions, we add the assumptions needed for comparative statics and for dynamic properties.

2.1 The Model Economy

In the model economy, there exist consumers with infinitely long lives and a government. All consumers have the same preference and are represented as a single agent. We assume no population growth and exogenous technological progress.

The representative agent with perfect foresight solves the following problem:

$$\max_{\{c_t, m_t\}} \int_0^\infty \mathrm{e}^{-\Delta_t} u(c_t, m_t) dt, \quad \text{s.t.}$$

$$\dot{a}_t = f(k_t) - \pi_t m_t + v_t - c_t, \tag{1}$$

$$\dot{\Delta}_t = \rho(c_t, m_t), \tag{2}$$

where $a_0 > 0$,[5] $\Delta_0 = 0$, c_t is consumption, m_t is the real money balance, k_t is capital, $a_t = k_t + m_t$ is an asset, π_t is the inflation rate, v_t is a lump-sum government transfer, f is the production function, u is the felicity function, ρ is the subjective discount rate function, and Δ is the cumulative subjective discount rate function. A higher ρ indicates an agent with higher marginal impatience.[6] This functional form includes the case of Uzawa (1968) by setting $\rho = \tilde{\rho}(u(c, m))$.

We assume that u, ρ, and f are twice continuously differentiable. We assume that $u_c \geq 0$, $u_m \geq 0$, $u_{cc} \leq 0$, $u_{mm} \leq 0$, $0 < \rho < \infty$ for all $c > 0$ and $m > 0$, and $f > 0$ for all $k > 0$. These are standard assumptions. Additional assumptions regarding u, ρ, and f are imposed after the steady state is defined.

The level of felicity u is often assumed to be negative in the field of literature that includes Epstein and Hynes (1983) and Obstfeld (1990). The class of felicity function taking a negative value covers the Epstein–Hynes function ($u = -1$) and the famous constant relative risk aversion (CRRA) functional form $(c^{1-\alpha} m^{\alpha})^{1-\sigma}/(1-\sigma)$ for $\sigma > 1$ and $0 < \alpha < 1$. Such negativity is a sufficient condition for the concavity of the Hamiltonian, defined later. On the other hand, some authors, including Uzawa (1968), Das (2003), and, implicitly, Chen et al. (2008), assume that $u > 0$. In the present paper, the level of felicity is allowed to be either positive or negative. However, the level of felicity cannot equal zero ($u \neq 0$) for all c, $m > 0$.

We do not impose any restrictions on the first degree of derivatives of ρ. Whereas $\rho_c > 0$ and $\rho_m > 0$ are referred to as IMI in consumption and money, respectively, $\rho_c < 0$ and $\rho_m < 0$ are referred to as DMI in consumption and money, respectively. Most authors, except for Das (2003), assumed IMI in a real economy for stability purposes, although they have admitted that IMI is not consistent with several studies in the empirical literature (Becker and Mulligan, 1997, for example). According to Chen et al. (2008), a reverse Tobin effect, i.e., a negative relationship between inflation and economic growth, emerges when $\rho_m < 0$ in the MIUF model, but they assumed the steady state to be locally stable. The present study investigates the additional conditions that are needed in the case of IMI, or whether DMI is justified from a theoretical viewpoint, and thus begins without restrictions on ρ_c and ρ_m.

In order to solve the above problem, we consider the following present value Hamiltonian:

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[5] Initial positive nominal money stock is also required for the finite growth rate of the money supply by the government at the next continuous period.

[6] Some authors, including Chen et al. (2008), use impatience instead of marginal impatience.

$$H = \mathrm{e}^{-\Delta}\{u(c,m) + \lambda(f(k) - \pi m + v - c) - \phi\rho(c,m) + \psi(a - m - k)\}, \quad (3)$$

where ψ denotes the Lagrange multiplier for $a = m + k$, and λ and ϕ denote the costate variables associated with (1) and (2), respectively. Applying the Pontryagin maximum principle yields the following first-order necessary conditions:

$$u_c - \lambda - \phi\rho_c = 0, \quad (4)$$

$$u_m - \lambda(f_k + \pi) - \phi\rho_m = 0, \quad (5)$$

$$\dot{\lambda} = \lambda(\rho - f_k)\,, \quad (6)$$

$$\dot{\phi} = -u + \phi\rho, \quad (7)$$

and the transversality conditions $\lim_{t\to\infty} \lambda_t\, a_t \mathrm{e}^{-\Delta_t} = 0$ and $\lim_{t\to\infty} \phi_t\, \Delta_t \mathrm{e}^{-\Delta_t} = 0$. Since $a = m + k$ is binding, $\psi = \lambda f'(k) > 0$.

For a sufficient condition for the maximization problem, we should assume that the Hamiltonian (3) is concave with respect to c, m, k, a, and Δ for any multipliers λ, ϕ, and ψ in a strict sense. One of the sufficient conditions is that $\mathrm{e}^{-\Delta}u(c,m)$, $f(k)$, and $\rho(c,m)$ are concave when the costate variables in the Hamiltonian are set to $\tilde{\phi} = -\mathrm{e}^{-\Delta}\phi$ and $\tilde{\lambda} = \mathrm{e}^{-\Delta}\lambda$. The concavity of $\mathrm{e}^{-\Delta}u(c,m)$ holds if $u < 0$, $u_{cc} < 0$, $u_{mm} < 0$, $u_{cc}u_{mm} - u_{cm}^2 > 0$, and $u(u_{cc}u_{mm} - u_{cm}^2) + u_c(u_m u_{cm} - u_{mm}u_c) + u_m(u_c u_{cm} - u_{cc}u_m) < 0$ for all c and m, excluding the possibility that $u > 0$. However, even if these conditions are not satisfied, the indirect utility function might be concave. Assuming the indirect utility or the value function to be concave, Chang (1994) adopted a dynamic programming approach and allowed the felicity to take positive and negative values. The present paper does not discuss the sufficient condition obtained from the Hamiltonian concavity.

The solution to (7) with $\lim_{t\to\infty} \phi\, \Delta \mathrm{e}^{-\Delta} = 0$ is:

$$\phi(t) = \int_t^\infty \mathrm{e}\,\mathrm{xp}\left\{-\int_t^v \rho\,(c_\tau, m_\tau)\mathrm{d}\tau\right\} u(c_v, m_v)\mathrm{d}v,$$

which is the lifetime utility from the period t. The value of ϕ is negative (resp. positive) when $u <$ (resp. $>$) 0. As Obstfeld (1990) and Chang (1994) have shown, $\phi(t)$ on the optimal path is the same as the solution to the following continuous version of the Bellman equation:

$$V(a) = \max_{\substack{c_t, m_t;\\ 0\le t\le \delta t}} \left\{\int_0^{\delta t} \mathrm{e}^{-\Delta_t}\, u(c_t, m_t)dt + \mathrm{e}^{-\Delta_{\delta t}} V(a + \delta a)\right\},$$

with the budget constraint (1) and $a = m + k$ for a small $\delta t > 0$. When $\delta t \to 0$, the Bellman equation is:

$$0 = \max_{c,m}\{\,u(c,m) - \rho(c,m)V(a) + \dot{a}V'(a)\}.$$

subject to (1) and $a = m + k$. The first-order condition with respect to c yields $V'(a) = u_c - \rho_c V(a)$. The marginal utility in consumption depends on the marginal felicity as well as the future lifetime utility through an endogenous time preference. Clearly, $\lambda = u_c - \rho_c\phi = V'(a)$. We assume that the lifetime utility is increasing in assets and that the shadow price of assets is positive.

We define the money demand function used for investigating the local stability of the dynamic system in the next subsection. Let $R_t = f_k + \pi_t$ represent nominal interest rates, the cost of holding money. We implicitly define the money demand function $\varphi(c,k,R)$[7] using

$$R_t = \frac{u_m(c,m) - V(m+k)\rho_m(c,m)}{u_c(c,m) - V(m+k)\rho_c(c,m)}. \tag{8}$$

The money demand function φ depends on nominal interest rates, consumption, and capital. In the case of a constant time preference or of the Uzawa-type discount rate, the right side of (8) is simplified to $u_m(c,m)/u_c(c,m)$, and, therefore, the term of capital is not augmented in the money demand function φ.

Now, we return to the remaining model components; the government and equilibrium conditions. The government prints money at a constant rate μ and runs a balanced budget by transferring seigniorage revenues to the consumers: $v_t = \mu m_t$. In equilibrium, the money and the goods markets are clear:

$$\dot{m} = (\mu - \pi)m, \tag{9}$$

$$\dot{k} = f(k) - c. \tag{10}$$

We define a steady state $(c^*, m^*, k^*, \lambda^*, \phi^*, \pi^*)$ when the variables satisfy (4), (5), (6), (7), (9), (10), and $\dot{\lambda} = \dot{\phi} = \dot{m} = \dot{k} = 0$.

We assume that there exists a steady state such that $c^* > 0$, $m^* > 0$, $k^* > 0$, $\lambda^* = u_c - \rho_c\phi^* = V'(m^* + k^*) > 0$, $\phi^* = u(c^*, m^*)/\rho(c^*, m^*) = V(m^* + k^*)$, and $\pi^* = \mu$. We assume that both real and nominal rates of interest are positive: $f_k > 0$ and $\mu + f_k > 0$. To establish (8) with $m^* > 0$, $\lambda > 0$, and $R > 0$, we must assume the positive marginal utility in real balances: $u_m - \rho_m\phi^* > 0$. The positive marginal utility in consumption and real balances is written as $u_c - u\rho_c/\rho > 0$ and $u_m - u\rho_m/\rho > 0$ for a steady state. Both conditions hold automatically when $u < 0$, $\rho_c \geq 0$, and $\rho_m \geq 0$.

Furthermore, we assume that $f_{kk} < 0$ and $u_{cc} - u\rho_{cc}/\rho < 0$ in a steady state. The former implies that the marginal productivity of capital is decreasing. The latter is used for a positive intertemporal elasticity of substitution. Similarly, we assume that $u_{mm} - u\rho_{mm}/\rho < 0$.

Chen et al. (2008) assumed that $u_c/u - \rho_c/\rho > 0$ and $u_{cc}/u - \rho_{cc}/\rho < 0$ for a steady state because they assumed implicitly that $u > 0$. When $u < 0$, the inequalities should be the opposite. In addition, Chen et al. assumed that the curvature of felicity is larger than that of the discount rate with respect to consumption. Note that our assumption that $u_{cc} - u\rho_{cc}/\rho < 0$ always holds when $u < 0$, $u_{cc} < 0$, and $\rho_{cc} < 0$.

We make an additional assumption with respect to the money demand function φ. We assume that the money demand function is decreasing in the nominal interest rates for a steady state. That is, a higher cost of holding money R_t reduces money demand. Let the right-hand side of (8) be $L = (u_m - V\rho_m)/(u_c - V\rho_c)$. The total differential is $\mathrm{d}R = L_m \mathrm{d}m + L_c \mathrm{d}c + L_k \mathrm{d}k$, where L_x for $x = c, m, k$, is shown in (40), (41), (42) of Appendix I. The implicit function theorem indicates that $\varphi_R = 1/L_m$, $\varphi_c = -L_c/L_m$ and $\varphi_k = -L_k/L_m$. A decreasing money demand for nominal interest rates ($\varphi_R < 0$) is equivalent to $L_m < 0$ for a steady state. We do not make any assumptions regarding the signs of φ_c and φ_k (L_c and L_k) at this stage.

Before closing this subsection, we summarize the assumptions on the felicity u, the discount rate ρ, the production technology f, and the money demand φ that we have made thus far.

1. $u \neq 0$, $u_c \geq 0$, $u_m \geq 0$, $u_{cc} \leq 0$, $u_{mm} \leq 0$ and $0 < \rho < \infty$ for all $c, m > 0$

[7] Here, a consumer behaves as a price taker and, accordingly, the money demand is a function of c, k, and R, even though R is a function of k in equilibrium. In addition, the explanation of $\varphi(c,k,R)$ is more intuitive than the explanation of $\tilde{\varphi}(c,k,\pi) = \varphi(c,k,f_k + \mu)$.

and $f > 0$ for all $k > 0$.

2. $f_k > 0$, $\mu + f_k > 0$, $f_{kk} < 0$, $u_c - u\rho_c/\rho > 0$, $u_m - u\rho_m/\rho > 0$, $u_{cc} - u\rho_{cc}/\rho < 0$, and $u_{mm} - u\rho_{mm}/\rho < 0$ for a steady state.
3. $\varphi_R < 0$ or, equivalently, $L_m < 0$ for a steady state.

Note that ρ_c and ρ_m may take negative values. Starting with these assumptions, we add the required conditions for the local stability of the dynamic system in the next subsection.

2.2 Local Stability Conditions

This subsection examines local stability conditions. For this purpose, we approximate the dynamic system around the steady state using the Jacobian matrix and then reproduce the result of Chen et al. (2008) in a three-dimensional system to confirm that the determinant of the Jacobian matrix should be negative in order to justify the comparative statics. We then examine the local stability of the dynamic system by considering the determinant and the trace of the matrix separately, which provides us with two propositions. Combining the two propositions yields the main results of this section as a single theorem. After examining the independence of money demand from capital, we present conditions for local stability, using only preferences and technology, and a corollary. Finally, we present two corollaries relating the present results to the results of previous studies.

Linearizing the dynamic system around the steady state yields:

$$\begin{bmatrix} \dot{c} \\ \dot{m} \\ \dot{k} \end{bmatrix} = J_{MIUF} \begin{bmatrix} c - c^* \\ m - m^* \\ k - k^* \end{bmatrix},$$

where J_{MIUF} is the Jacobian matrix of the dynamic system, specified in (43) of Appendix I.

Before analyzing the local stability conditions, we mention the comparative statics conducted by Chen et al. (2008). The final part of Appendix I provides the following results:

$$\begin{bmatrix} \mathrm{d}c^*/\mathrm{d}\mu \\ \mathrm{d}m^*/\mathrm{d}\mu \\ \mathrm{d}k^*/\mathrm{d}\mu \end{bmatrix} = \frac{c^* m^*}{\theta\, det\,(J_{MIUF})} \begin{bmatrix} -\rho_m f_k \\ -(f_{kk} - \rho_c f_k) \\ -\rho_m \end{bmatrix}, \tag{11}$$

where $\theta = -(u_{cc} - \rho_{cc}V)c^*/(u_c - \rho_c V)$ is the intertemporal elasticity of substitution. By assumption, $\theta > 0$. The correlation between capital stocks and inflation depends only on the sign of ρ_m. That is, as long as $\det(J_{MIUF})$ is negative, $\mathrm{d}k^*/\mathrm{d}\mu >$ (resp. $<$) 0 if $\rho_m >$ (resp. $<$) 0. Our results are the same as those provided in Table 1 in Chen et al., although they assume that the Jacobian matrix of the four-dimensional dynamic system has a negative determinant.

Next, we investigate the dynamic system. In the MIUF model, consumption, c, and real money balances, m, are jump variables. The consumer can choose any values of c and m. Therefore, we require only one negative root in the three-dimensional system for stability in terms of economic theory. As sufficient conditions, the determinant $\det(J_{MIUF})$ should be negative and the trace $\mathrm{tr}(J_{MIUF})$ should be positive. In what follows, we examine the conditions for $\det(J_{MIUF}) < 0$ and $\mathrm{tr}(J_{MIUF}) > 0$ separately and then combine these conditions to obtain the main results in the theorem.

First, we present conditions for $\det(J_{MIUF}) < 0$. By performing a few algebraic operations (see Appendix I), we have:

$$\det(J_{MIUF}) = \frac{c^* m^*}{\theta}\{\rho_m(f_{kk} - L_k - f_k L_c) + L_m(f_k \rho_c - f_{kk})\}, \tag{12}$$

where L_x for $x = c, m, k$, is defined in the previous subsection, and each concrete expression is given in Appendix I. Since $c^* > 0$, $m^* > 0$, $k^* > 0$, $\theta > 0$ and $L_m < 0$, by assumption, we can say that if

$$\rho_m (f_{kk} - L_k - f_k L_c)/L_m + f_k \rho_c - f_{kk} > 0 \tag{13}$$

holds, then $\det(J_{MIUF}) < 0$.

This condition is interpreted as a monetary version of the BG condition (Brock and Gale, 1969), in which the increase in the discount rate dominates the increase in the marginal product of capital ($f_{kk} < 0$) for a steady state. Chen et al. (2008) assumed that $f_k \rho_c - f_{kk} > 0$[8] and referred to this assumption as the CP condition. Since, in a steady state, $f_k = \mathrm{d}c/\mathrm{d}k$, the CP condition indicates the dominance of the increase in the discount rate over the increase in the marginal product of capital for a steady state, ignoring the effect of money on the degree of impatience. As a monetary version of the CP condition or the BG condition, we instead propose the condition (13).

To justify this proposal, we first consider the meaning of $(f_{kk} - L_k - f_k L_c)/L_m$ in (13). Remember that the money demand function $\varphi(R, c, k)$ is implicitly defined as (8), and the total differential of $L = (u_m - \rho_m V(m+k))/(u_c - \rho_c V(m+k))$ is $\mathrm{d}R = L_m \mathrm{d}m + L_c \mathrm{d}c + L_k \mathrm{d}k$. Then, using the implicit function theorem, we can show that:

$$\frac{\mathrm{d}m}{\mathrm{d}k} = \varphi_R \frac{\mathrm{d}R}{\mathrm{d}k} + \varphi_c \frac{\mathrm{d}c}{\mathrm{d}k} + \varphi_k = \frac{1}{L_m}\left\{\frac{\mathrm{d}R}{\mathrm{d}k} - L_c \frac{\mathrm{d}c}{\mathrm{d}k} - L_k\right\} = \frac{f_{kk} - L_k - f_k L_c}{L_m}.$$

Note that $R = f_k + \mu$ and $\mathrm{d}R/\mathrm{d}k = f_{kk}$ at a steady state. Thus, $(f_{kk} - L_k - f_k L_c)/L_m$ indicates the degree to which capital increases real balances under a steady-state equilibrium. Adding the direct effect of capital on consumption, $\mathrm{d}c/\mathrm{d}k$, we can obtain a monetary version of the BG condition, which is $\rho_m \mathrm{d}m/\mathrm{d}k + \rho_c \mathrm{d}c/\mathrm{d}k - f_{kk} > 0$. Since $c^* > 0$, $m^* > 0$, $k^* > 0$, $\theta > 0$ and $\varphi_R < 0$, by assumption, the BG condition is equivalent to a negative determinant of the matrix.

The above discussion can be summarized as the following proposition:

Proposition 1. *Suppose that the assumptions in the previous subsection hold. If the BG condition (13) holds, then* $det\,(J_{MIUF}) < 0$.

Although the BG condition is satisfied when $\rho_m \geq 0$, $\rho_c \geq 0$, $L_k \geq 0$ and $L_c \geq 0$, it does not eliminate the possibility of a negative ρ_m for a steady state. A sufficiently high curvature of f and a sufficiently inelastic[9] money demand function with respect to c, k, and R may lead to a reverse Tobin effect. As long as the BG condition holds, the comparative static analysis by Chen et al. (2008) is justified.

Next, we provide the conditions for $\mathrm{tr}(J_{MIUF}) > 0$. In Appendix I, $\mathrm{tr}(J_{MIUF}) = -mL_m + f_k$ for a steady state if $L_c = 0$, and

$$\begin{aligned}\mathrm{tr}(J_{MIUF}) = {} & \frac{c^* m^*}{\theta} \rho_c L_c - m^* L_k + f_k \\ & + m^* \frac{(u_{cm} - u\rho_{cm}/\rho)^2 - (u_{cc} - u\rho_{cc}/\rho)(u_{mm} - u\rho_{mm}/\rho)}{(u_c - u\rho_c/\rho)(u_{cc} - u\rho_{cc}/\rho)}\end{aligned} \tag{14}$$

for a steady state if $L_c \neq 0$.

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[8] As discussed in the introduction, Chen et al. (2008) used this assumption simply to prove the unique existence of the steady state under the special case of $\rho_m = 0$.

[9] Specifically, (13) is written as: $f_{kk}(\rho_m \varphi_R - 1) + \rho_m \varphi_k + \rho_m f_k \varphi_c + f_k \rho_c > 0$.When $\rho_m \varphi_R < 1$, a larger $|f_{kk}|$ and a smaller $|\varphi_R|$, $|\varphi_c|$, and $|\varphi_k|$ make the BG condition attainable.

As discussed earlier, $\varphi_R = 1/L_m$, $\varphi_c = -L_c/L_m$ and $\varphi_k = -L_k/L_m$, and we have assumed that increasing the cost of holding money lowers money demand ($\varphi_R < 0$). Therefore, when $L_c = 0$ or $\varphi_c = 0$, the trace is always positive. Consider the case in which $\varphi_c > 0$. Then, if

$$\left(u_{cc} - u\frac{\rho_{cc}}{\rho}\right)\left(u_{mm} - u\frac{\rho_{mm}}{\rho}\right) - \left(u_{cm} - u\frac{\rho_{cm}}{\rho}\right)^2 \geq 0, \tag{15}$$

and $c^* m^* \rho_c L_c/\theta - m^* L_k + f_k > 0$, then the trace is positive.

As a sufficient condition, we can state that $c^* m^* \rho_c L_c/\theta - m^* L_k + f_k > 0$, if $\rho_c \geq 0$, $L_c \geq 0$, and $L_k \leq 0$. That is, when the discounting rate is non-decreasing in consumption, and money demand is increasing in consumption and non-decreasing in capital, the trace is positive.

We can obtain the following proposition:

Proposition 2. *Suppose that the assumptions in the previous subsection hold.*

1. *If the money demand function is independent of consumption, then* $tr(J_{MIUF}) > 0$.
2. *If* $\rho_c \geq 0$*, condition (15) holds, and the money demand function is non-decreasing in consumption and non-increasing in capital, then* $tr(J_{MIUF}) > 0$.

We have two comments regarding the proposition. First, condition (15) is established *either* when u and ρ are concave with respect to c and m in the case of $u < 0$ *or* when u is concave and ρ is convex in the case of $u > 0$. This is because a linear combination of concave functions ($u(c,m) - \rho(c,m)u(c^*,m^*)/\rho(c^*,m^*)$ for fixed c^* and m^*) is also a concave function. Thus, in the case of $u < 0$, the stronger the concavity of the felicity function and the discounting function, the higher the possibility of DMI in consumption. On the other hand, in the case of $u > 0$, if the concavity of the felicity function and the convexity of the discounting function become stronger, then DMI in consumption is more plausible.

Second, since $\varphi_R = 1/L_m < 0$, $\lambda = u_c - \rho_c V(m+k) > 0$, and L_k is specified in Equation (40) in Appendix I, non-increasing money demand in capital is equivalent to:

$$\rho_c u_m - \rho_m u_c \leq 0, \tag{16}$$

for a steady state. Note that equality is established in a constant preference case ($\rho_c = \rho_m = 0$), in the Epstein–Hynes utility function case ($u = -1$), or in the Uzawa case ($\rho = \tilde{\rho}(u(c,m))$).

Condition (16) sheds new light on the comparative statics conducted by Chen et al. (2008). They claimed that a reverse Tobin effect emerges in the case of $\rho_m < 0$, independent of ρ_c, when they assumed the determinant of the Jacobian matrix to be negative. On the other hand, condition (16) states that a negative ρ_m involves a negative ρ_c in the case of a positive marginal felicity. Furthermore, as claimed in Proposition 2, the steady state might be unstable for the case in which $\rho_c < 0$. Unlike Chen et al., we claim that a combination of $\rho_m < 0$ and $\rho_c \geq 0$ is less plausible from a theoretical viewpoint.

We combine the two propositions to present conditions for local stability. A sufficient condition for local stability is both a negative determinant and a positive trace of the Jacobian matrix.

Theorem 1. *Suppose that the assumptions in the previous subsection are satisfied.*

1. *If conditions (13) and (15) hold, the money demand function is increasing in consumption and non-increasing in capital (*$\varphi_c > 0$, $\rho_c u_m \leq \rho_m u_c$*), and* $\rho_c \geq 0$ *for a steady state, then the steady state is locally stable.*
2. *If condition (15) holds, the money demand function is increasing in consumption*

and independent of capital ($\varphi_c > 0$, $\rho_c u_m = \rho_m u_c$)*, and* $\rho_c \geq 0$ *for a steady state, then the steady state is locally stable.*

3. *If condition (13) holds and the money demand function is independent of consumption* ($\varphi_c = 0$)*, then the steady state is locally stable.*

The first two statements imply that the steady state leading to a Tobin effect is locally stable. However, this does not preclude DMI from resulting in a reverse Tobin effect. Decreasing marginal impatience in consumption and real balances is more plausible in conditions in which the concavity of the production function becomes stronger, the money demand function becomes less elastic, and the curvatures of the felicity and the discount rate functions become stronger.

Condition $u_c\rho_m = u_m\rho_c$ in the second statement is obtained as follows. A sufficient condition for a negative determinant or the BG condition (13) is that $\rho_c \geq 0$, $\rho_m \geq 0$, $\varphi_c \geq 0$, and $\varphi_k \geq 0$. On the other hand, Proposition 2 requires $\rho_c \geq 0$, $\varphi_c \geq 0$ and $\varphi_k \leq 0$ for a positive trace of the Jacobian matrix. Thus, when considering the sufficient condition for local stability, the money demand function should be independent of capital ($\varphi_k = 0$).

Condition $u_c\rho_m = u_m\rho_c$ ($L_k = 0$) makes the representation of L_m and L_c simpler, and gives more specific sufficient conditions for local stability. As long as u_c, u_m, ρ_c, and ρ_m never take a value of zero, then $L_m < 0$, $L_c \geq 0$ and (15) are represented as *either* $L_m = (u_c u_{mm} - u_m u_{cm})/u_c^2 < 0$ and $L_c = (u_c u_{cm} - u_m u_{cc})/u_c^2 \geq 0$ *or* $(\rho_c\rho_{mm} - \rho_m\rho_{cm})/\rho_c^2 < 0$ and $(\rho_c\rho_{cm} - \rho_m\rho_{cc})/\rho_c^2 \geq 0$. The former set of conditions corresponds to the case of Fischer (1979), whereas the latter corresponds to the case of Epstein and Hynes (1983). A sufficient condition for the former is $u_{cm} \geq 0$ (Calvo, 1979), whereas a sufficient condition for the latter is $\rho_{cm} \geq 0$. Since we would like to obtain the detailed condition of ρ_c or ρ_m for local stability, we consider only the former representation. Consequently, under $u_c/\rho_c = u_m/\rho_m$, the decreasing money demand in nominal interest rates and the increasing money demand in consumption are equivalent to $u_c u_{mm} - u_m u_{cm} < 0$ and $u_c u_{cm} - u_m u_{cc} > 0$.

Using the above representations, the BG condition (13) is satisfied when:

$$\rho_c[f_{kk}u_c^2 u_m - f_k(2u_c u_m u_{cm} - u_m^2 u_{cc} - u_c^2 u_{mm})] - f_{kk}(u_c^2 u_{mm} - u_c u_m u_{cm}) < 0, \tag{17}$$

and the trace is positive if (as a sufficient condition):

$$\frac{(u_{cc}/u - \rho_{cc}/\rho)(u_{mm}/u - \rho_{mm}/\rho) - (u_{cm}/u - \rho_{cm}/\rho)^2}{(u_c/u - \rho_c/\rho)^2} + \frac{\rho_c}{u_c^2}(u_{cm}u_c - u_{cc}u_m) > 0 \tag{18}$$

for a steady state. Both conditions are satisfied when $\rho_c \geq 0$ and *either* u and ρ are concave in the case of $u < 0$ *or* u is concave and ρ is convex in the case of $u > 0$. In condition (18), it remains difficult to provide an explicit expression of ρ_c. However, it is much easier to check numerically whether there exists a negative ρ_c, such as in (17) and (18), as compared to the case in which $\rho_c u_m \neq \rho_m u_c$.

The third statement in Theorem 1 implies that if the money demand is a function only of nominal interest rates ($u_c u_{cm} - u_m u_{cc} = 0$, $\rho_c u_m = \rho_m u_c$), then condition (18) is no longer needed, and, accordingly, condition (17) reduces to

$$\rho_c > \frac{f_{kk}(u_c u_{mm} - u_m u_{cm})}{f_{kk}u_c u_m + f_k(u_c u_{mm} - u_m u_{cm})},$$

for a steady state. That is, ρ_c can be expressed explicitly in such a special case.

Assuming $u_c\rho_m = u_m\rho_c$ ($\varphi_k = 0$) is also helpful for comparison with previous literature. We present two corollaries.

Corollary 1. *Suppose that the first two assumptions in the previous subsection are satisfied. Furthermore, we assume that $\rho_c u_m = \rho_m u_c$ and u is concave. If $\rho_c \geq 0$, either $u < 0$ and ρ is concave or $u > 0$ and ρ is convex, and either $u_c u_{mm} - u_m u_{cm} < 0$ and $u_c u_{cm} - u_m u_{cc} > 0$ or $u_{cm} > 0$ for a steady state, then the steady state is locally stable.*

This is consistent with Uzawa (1968) for the case in which $u > 0$, $\rho_c > 0$, $\rho_{cc} > 0$, and $\rho_c u_m = \rho_m u_c$, and with Fischer (1979) and Calvo (1979) for the case of a constant time preference ($\rho_c = \rho_m = 0$).

We can easily derive an impatience representation.

Corollary 2. *Suppose that the first two assumptions in the previous subsection are satisfied. If $\rho_c > 0$, $\rho_c u_m = \rho_m u_c$, and u is concave, either $u < 0$ and ρ is concave or $u > 0$ and ρ is convex, and either $\rho_c\rho_{mm} - \rho_m\rho_{cm} < 0$ and $\rho_c\rho_{cm} - \rho_m\rho_{cc} > 0$ or $\rho_{cm} > 0$ for a steady state, then the steady state is locally stable.*

This is consistent with Epstein and Hynes (1983) assuming $u = -1$. Note that, in either case, ρ_c or ρ_{cc} should be selected so that $u_c - u\rho_c/\rho > 0$ and $u_{cc} - u\rho_{cc}/\rho < 0$.

3. The TC Model

This section deals with the TC model, in which purchasing consumption goods involves transactions costs. Money is demanded because money reduces these costs. The present paper uses the model structures and most of the notation in Chen et al. (2008). This section follows the same order of discussion developed in the previous section. The first subsection describes the model economy and then defines a steady state, and the second subsection examines the local dynamic properties. Certain explanations are omitted to avoid repetition.

3.1 The Model Economy

As in the MIUF model, the economy has a constant population and no technological progress and involves homogeneous consumers and a government. A representative agent with perfect foresight solves the following problem:

$$\max_{\{c_t, m_t\}} \int_0^\infty \mathrm{e}^{-\Delta_t} u(c_t)dt, \quad \text{s.t.}$$

$$\dot{a}_t = f(k_t) - \pi_t m_t + v_t - c_t - T(c_t, m_t), \tag{19}$$

$$\dot{\Delta}_t = \rho(c_t), \tag{20}$$

where $a_0 > 0$ is given, T is the transaction costs function, and c, m, k, a, π, v, u, ρ, f, and Δ are as described in the previous section. We assume that u, ρ, f, and T are twice continuously differentiable. We assume that $u \neq 0$, $u_c \geq 0$, $u_{cc} \leq 0$, and $0 < \rho < \infty$ for all $c > 0$, and $f > 0$ for all $k > 0$. The conditions of u, ρ, and f are the same as in the MIUF model, except that u and ρ are functions of consumption only. We assume that $T > 0$, $T_c > 0$, $T_m < 0$ for $c, m > 0$. There exist positive transaction costs for positive consumption. Such costs are raised by consumption but reduced by real balances. Other assumptions regarding u, ρ, f, and T are imposed later.

Let the present-value Hamiltonian be:

$$H = \mathrm{e}^{-\Delta}\{u(c) + \lambda(f(k) - \pi m + v - c - T(c,m)) \\ -\phi\rho(c) + \psi(a - m - k)\}, \quad (21)$$

where ψ is the multiplier, and λ and ϕ denote the costate variable associated with (19) and (20). The first-order necessary conditions are:

$$u_c - \lambda(1 + T_c) - \phi\rho_c = 0, \quad (22)$$

$$\lambda(f_k + \pi + T_m) = 0, \quad (23)$$

(6), (7), $\lim_{t\to\infty} \lambda_t\, a_t \mathrm{e}^{-\Delta_t} = 0$ and $\lim_{t\to\infty} \phi_t\, \Delta_t \mathrm{e}^{-\Delta_t} = 0$. As in the previous section, ϕ_t, the lifetime utility from the period t, is equal to $V(a_t)$, and λ_t is equal to $V'(a_t)$.

The government acts in the manner described in the previous section. In equilibrium, the money market condition is unaltered (9) but the goods market condition is:

$$\dot{k} = f(k) - c - T(c,m). \quad (24)$$

A steady state is defined as in the previous section, and we assume that there exists a steady state such that: $c^* > 0$, $m^* > 0$, $k^* > 0$, $\lambda^* = (u_c - \phi^*\rho_c)/(1 + T_c) > 0$, $\phi^* = u(c^*)/\rho(c^*)$, $f_k > 0$, and $\pi^* + f_k > 0$. As $T_c > 0$, we need $u_c - \rho_c u/\rho > 0$ for $\lambda^* > 0$. This is satisfied when $u < 0$ and $\rho_c > 0$.

We assume that $u_{cc} - \rho_{cc}u/\rho < 0$ and $T_{cc} > 0$ for a steady state. These are the sufficient conditions for the positiveness of the intertemporal elasticity of substitution. The former condition is established if either $\rho_{cc} > 0$ and $u > 0$ or $\rho_{cc} < 0$ and $u < 0$.

As in the previous section, we define the money demand function φ from (23) as $R + T_m(c,m) = 0$ implicitly. Unlike in the MIUF model, the money demand function is independent of capital, and is characterized not by preferences (u and ρ), but by the transaction cost technology T only. The total differential is $\mathrm{d}R + T_{mm}\mathrm{d}m + T_{cm}\mathrm{d}c = 0$. The implicit function theorem implies that $\varphi_R = -1/T_{mm}$. We assume that $0 < T_{mm} < \infty$ for $-\infty < \varphi_R < 0$. We can also demonstrate that $\varphi_c = -T_{cm}/T_{mm}$. The sign of T_{cm} determines whether money demand is increasing in consumption, but we provide a restriction on T_{cm} in the next subsection.

Next, we summarize the assumptions regarding u, ρ, f, T, and φ that we have made thus far.

1. $u \neq 0$, $u_c \geq 0$, $u_{cc} \leq 0$, and $0 < \rho < \infty$ for all $c > 0$, and $f > 0$ for all $k > 0$.
2. $f_k > 0$, $\mu + f_k > 0$, $f_{kk} < 0$, $u_c - u\rho_c/\rho > 0$, and $u_{cc} - u\rho_{cc}/\rho < 0$ for a steady state.
3. $T > 0$, $T_c > 0$, $T_m < 0$ for all c and $m > 0$, and $T_{cc} > 0$ and $0 < T_{mm} < \infty$, or equivalently $-\infty < \varphi_R < 0$, for a steady state.

3.2 Local Stability Conditions

In this subsection, as in the previous section, we examine the local stability conditions. We provide a linear approximation of the dynamic system around the steady state using the Jacobian matrix and reproduce the comparative statics of Chen et al. (2008). We then present the condition for a negative determinant and a positive trace of the matrix as two propositions. In addition, we provide a theorem combining the two propositions.

Appendix II gives a linearized dynamic system around the steady state:

$$\begin{bmatrix} \dot{c} \\ \dot{m} \\ \dot{k} \end{bmatrix} = J_{TC} \begin{bmatrix} c - c^* \\ m - m^* \\ k - k^* \end{bmatrix},$$

where J_{TC} is specified in (47) of Appendix II. In addition, Appendix II conducts a comparative static analysis with respect to the rate of money growth:

$$\begin{bmatrix} \mathrm{d}c^*/\mathrm{d}\mu \\ \mathrm{d}m^*/\mathrm{d}\mu \\ \mathrm{d}k^*/\mathrm{d}\mu \end{bmatrix} = \frac{c^* m^*}{\theta\, det\,(J_{TC})} \begin{bmatrix} T_m f_k \\ \rho_c f_k - (1 + T_c) f_{kk} \\ -\rho_c f_k \end{bmatrix}, \tag{25}$$

where $\theta = \tilde{\theta} + cT_{cc}/(1 + T_c)$ and $\tilde{\theta} = -c(u_{cc} - \rho_{cc}V)/(u_c - \rho_c V)$ represents the intertemporal elasticity of substitution. Note that, by assumption, $\theta > 0$. Assuming that $\det\,(J_{TC}) < 0$, the monetary expansion effect is the same as in Table 1 in Chen et al. (2008). The TC model also predicts a positive (resp. negative) correlation when impatience increases (resp. decreases) with consumption.

Let us examine the dynamic system. Like the MIUF model, the TC model requires only one negative root in the three-dimensional system from the viewpoint of economic theory. As sufficient conditions, the determinant $\det\,(J_{TC})$ should be negative and the trace $\mathrm{tr}(J_{TC})$ should be positive. Similarly to Section 2.2, Section 3.2 examines the conditions for $\det\,(J_{TC}) < 0$ and $\mathrm{tr}(J_{TC}) > 0$ separately. These conditions are then combined in order to obtain the main result in this section as a theorem in conjunction with a corollary.

First, we present conditions for $\det\,(J_{TC}) < 0$. By performing a few algebraic operations (see Appendix II), we have:

$$\det\,(J_{TC}) = \frac{c^* m^*}{\theta} \{f_{kk}(T_{mm}(1 + T_c) - T_{cm}T_m) - \rho_c(T_{mm}f_k + T_m f_{kk})\}. \tag{26}$$

Note that, by assumption, $\theta > 0$. Thus, if:

$$\rho_c > \frac{f_{kk}\{(1 + T_c) - T_{cm}T_m/T_{mm}\}}{f_k + f_{kk}T_m/T_{mm}}, \tag{27}$$

then $\det\,(J_{TC}) < 0$.

As long as $(1 + T_c) - T_{cm}T_m/T_{mm} > 0$, then $\rho_c < 0$ and $\det(J_{TC}) < 0$ for a suitable choice of function ρ. This condition is required in the case of the TC model with a constant time preference ($\rho_c = 0$). To interpret $(1 + T_c) - T_{cm}T_m/T_{mm} > 0$, we use the money demand function φ. Since φ is defined as $R + T_m(c, \varphi(c, R)) = 0$ and $\varphi_c = -T_{cm}/T_{mm}$, the condition is equivalent to $1 + T_c + T_m\varphi_c = \mathrm{d}(c + T)/\mathrm{d}c > 0$. The differentiation indicates that, if $T_{cm} \geq 0$, then a direct increase in consumption raises transaction costs, whereas an indirect increase in money via the money demand function reduces transaction costs. Thus, the above condition is satisfied *either* if money demand decreases with consumption *or* if the marginal increase in consumption is not very large. When $c + T$ is referred to as "gross" consumption, we can say that an increase in "net" consumption strictly raises "gross" consumption for a given nominal rate of interest. This is more likely to hold for a smaller $|T_{cm}|$ and a larger $|T_{mm}|$, implying that the money demand function is less elastic with respect to consumption and nominal interest rates.

As in the MIUF model, (27) is a monetary version of the BG condition. To demonstrate this, assume that $(1 + T_c) - T_{cm}T_m/T_{mm} > 0$. Then, (27) can be rewritten as:

$$f_{kk} < \rho_c \frac{f_k + f_{kk} T_m \varphi_R}{1 + T_c + T_m \varphi_c} = \rho_c \frac{\mathrm{d}(c+T)/\mathrm{d}k}{\mathrm{d}(c+T)/\mathrm{d}c} = \rho_c \frac{\mathrm{d}c}{\mathrm{d}k}.$$

Note that $R = \mu + f_k$ and $c + T = f(k)$ in a steady state. The effect of capital on "net" consumption is described via "gross" consumption. Condition (27) means that the increase in the discount rate dominates the increase in the marginal product of capital for a steady state.

We can summarize the above discussion as the following proposition:

Proposition 3. *Suppose that the assumptions in the previous subsection hold. If the BG condition (27) holds, then* $det\ (J_{TC}) < 0$.

Like the MIUF model, the TC model requires the dominance of the discount rate over the marginal product of capital for $\det\ (J_{TC}) < 0$. The right-hand side of (27) is described only by the technology functions f and T and is independent of the preference function u or ρ. A negative value of ρ_c, leading to a reverse Tobin effect, is possible as long as an increase in "net" consumption strictly raises "gross" consumption for a given nominal rate of interest ($(1 + T_c) - T_{cm} T_m / T_{mm} > 0$) and becomes increasingly plausible as the curvature of the production function becomes higher and the money demand function becomes less elastic with respect to consumption and nominal interest rates. Even though a comparative static analysis is justified, we need to investigate $\mathrm{tr}(J_{TC}) > 0$ for local stability.

Next, we examine the conditions for $\mathrm{tr}(J_{TC}) > 0$. Appendix II demonstrates that $\mathrm{tr}(J_{TC}) = m^* T_{mm} + f_k$ for a steady state when $T_{cm} = 0$, and

$$\mathrm{tr}(J_{TC}) = \frac{(1+T_c)(u_{cc} - \rho_{cc} u/\rho) T_{mm}}{(1+T_c)(u_{cc} - \rho_{cc} u/\rho) - (u_c - \rho_c u/\rho) T_{cc}} m^* - \frac{(u_c - \rho_c u/\rho)(T_{cc} T_{mm} - T_{cm}^2 - \rho_c T_{cm})}{(1+T_c)(u_{cc} - \rho_{cc} u/\rho) - (u_c - \rho_c u/\rho) T_{cc}} m^* + f_k, \qquad (28)$$

for a steady state when $T_{cm} \neq 0$.

As discussed earlier, from the implicit function theorem, $\varphi_c = -T_{cm}/T_{mm}$. When money demand is independent of consumption ($\varphi_c = 0$), the trace is always positive because $\varphi_R = -1/T_{mm} < 0$ or $T_{mm} > 0$, by assumption. In addition, we have assumed that $T_c > 0$, $T_{cc} > 0$, $f_k > 0$, $u_c - u\rho_c/\rho > 0$, and $u_{cc} - u\rho_{cc}/\rho < 0$ for a steady state in the previous subsection. Even in the case of increasing money demand ($T_{cm} < 0$), the trace is positive if $T_{cc} T_{mm} - T_{cm}^2 - \rho_c T_{cm} \geq 0$.

Thus, we obtain the following proposition:

Proposition 4. *Assume that the assumptions in the previous subsection hold.*

1. *If the money demand function is independent of consumption (*$T_{cm} = 0$*), then* $tr(J_{TC}) > 0$.
2. *If the money demand function is increasing in consumption (*$T_{cm} < 0$*) and* $\rho_c \geq (T_{cc} T_{mm} - T_{cm}^2)/T_{cm}$*, then* $tr(J_{TC}) > 0$.

In order for ρ_c to take a negative value at the steady state, $T_{cc} T_{mm} - T_{cm}^2 > 0$, which holds when T is convex.[10] As long as the transaction cost function is strongly convex and the money

[10] We have already assumed that $T_{cc} > 0$ and $T_{mm} > 0$.

demand function is less elastic with respect to consumption, then there is potential for DMI in consumption.

As discussed in Propositions 3 and 4, we require that $(1+T_c)-T_{cm}T_m/T_{mm}>0$ and $T_{cc}T_{mm}-T_{cm}^2>0$ for ρ_c containing zero. Combining Propositions 3 and 4 gives the following theorem:

Theorem 2. *Assume that the assumptions in the previous subsection hold.*

1. *If the money demand function is increasing in consumption ($T_{cm}<0$), and*
$$\rho_c \geq max\left[\frac{f_{kk}\{(1+T_c)-T_{cm}T_m/T_{mm}\}}{f_k+f_{kk}T_m/T_{mm}}, \frac{T_{cc}T_{mm}-T_{cm}^2}{T_{cm}}\right]$$
for a steady state, then the steady state is locally stable.
2. *If $T_{cm}<0$, T is convex, and an increase in "net" consumption is additionally assumed to raise "gross" consumption for given nominal interest rates, then the steady state is locally stable for a negative ρ_c.*
3. *If $T_{cm}=0$, then the steady state is locally stable for*
$$\rho_c \geq \frac{f_{kk}(1+T_c)}{f_k+f_{kk}T_m/T_{mm}}.$$

Unlike in the case of the MIUF model, we can determine the range of ρ_c explicitly using only the production function f and the transaction costs technology T. Once the steady state is obtained, checking whether the comparative statics with DMI is justified and whether the steady state is stable is much easier than in the MIUF model.

As shown, the BG condition with DMI is more likely to hold as the production function becomes more concave and as the money demand function becomes less elastic, or if there is a small $|T_{cm}|$ and a large $|T_{mm}|$. As $|T_{cm}|$ becomes smaller and $|T_{mm}|$ becomes larger, the transaction costs function becomes more convex, making a positive trace of the Jacobian matrix more plausible. When holding money is relatively costless and ineffectual in making consumption easier, local stability under DMI is achievable.

4. Uniqueness of the Steady State

This section examines the conditions for the uniqueness of a steady state in the MIUF model and the TC model. As discussed earlier, the local stability conditions for both monetary models are treated in a united manner by using a monetary BG condition and the money demand function. This BG condition has played an essential role in determining the determinant of the Jacobian matrix. As conjectured in the introduction, if the BG condition is satisfied globally, then this condition should serve as the unique steady state for both monetary models. Below we demonstrate that this conjecture is correct.

Consider the MIUF model. The steady state (c^*, m^*, k^*) solves as:

$$\rho(c,m)=f_k(k) \tag{29}$$

$$c=f(k) \tag{30}$$

$$\frac{u_m-\rho_m u/\rho}{u_c-\rho_c u/\rho}=f_k+\mu. \tag{31}$$

Assume that the money demand function $\varphi(c,k,R)$ with $R=f_k+\mu$ is defined from (31) for all $c > 0$, $k > 0$, and $R > 0$. Substituting φ and (30) into (29) yields:

$$\rho(f(k), \varphi(f(k), k, f_k + \mu)) - f_k(k) = 0. \tag{32}$$

If there exists a unique k^* satisfying (32), then $c^* = f(k^*)$ and $m^* = \varphi(f(k^*), k^*, f_k^* + \mu)$.

We impose the standard Inada conditions on the production function: $f(0) = 0$, $f_k > 0$ and $f_{kk} < 0$ for all k, and $\lim_{k\to 0} f_k = \infty$ and $\lim_{k\to\infty} f_k = 0$. Furthermore, we assume that the discounting function has upper and lower bounds: $\rho^l \leq \rho \leq \rho^u$ for all c and m and $\rho^u > \rho^l > 0$.[11] The left-hand side of (32) is then smaller (resp. larger) than zero when $k \to 0$ (resp. $k \to \infty$). If the left-hand side is monotonically increasing, then there exists a unique k^*. For a monotonically increasing ρ, the derivative of the left-hand side of (32) with respect to k should be positive. In this case, the BG condition (13) is satisfied for all c, m, and k.

A similar argument applies to the TC model. The steady state (c^*, m^*, k^*) solves as:

$$\rho(c) = f_k(k) \tag{33}$$

$$c = f(k) - T(c, m) \tag{34}$$

$$-T_m = f_k + \mu. \tag{35}$$

Assume that the money demand function $\varphi(c, R)$ with $R = f_k + \mu$ is defined by (35) for all $c > 0$, $k > 0$, and $R > 0$.

Let 'gross' consumption be $x = c + T(c, \varphi(c, f_k + \mu))$. We assume that gross consumption is increasing in net consumption for a given k or $\partial x/\partial c > 0$ for all c and a given k. We can then obtain an inverse function, denoted by $c = \xi(x, k)$, for each k. Note that $\partial\xi/\partial x = 1/(\partial x/\partial c) = 1/(1 + T_c + T_m\varphi_c)$ and $\partial\xi/\partial k = (\partial x/\partial k)/(\partial x/\partial c) = f_{kk}T_m\varphi_R/(1 + T_c + T_m\varphi_c)$. It follows from (34) that $c = \xi(f(k), k)$. Substituting $c = \xi(f(k), k)$ into (33) yields:

$$\rho(\xi(f(k), k)) - f_k(k) = 0. \tag{36}$$

Like the MIUF model, the TC model assumes the Inada condition and the bounded discounting function. The condition for the uniqueness of the steady state is a positive derivative of the left-hand side of (36) with respect to k:

$$\begin{aligned}\rho_c \frac{\partial \xi}{\partial k} - f_{kk} &= \rho_c(\xi_x f_k + \xi_k) - f_{kk} \\ &= \rho_c \frac{f_k + f_{kk}T_m\varphi_R}{1 + T_c + T_m\varphi_c} - f_{kk} > 0\end{aligned}$$

for all k. This is satisfied if the BG condition (27) is satisfied globally.

Thus, we can obtain the following theorem:

Theorem 3. *When the money demand function is well defined globally and the Brock-Gale condition is satisfied globally,* $f(0) = 0$, $f_k > 0$, *and* $f_{kk} < 0$ *for all* k, *and* $lim_{k\to 0} f_k = \infty$ *and* $lim_{k\to\infty} f_k = 0$, *then the steady state exists uniquely in the MIUF model. If gross consumption is also increasing in net consumption for any* k *in the TC model, then there exists a unique steady state.*

For the unique steady state, both models require the BG condition as well as the money demand function. We present an example for each model. In the case of the MIUF model, we consider the utility function $u(c, m) = (c^{1-\alpha}m^{\alpha})^{1-\sigma}/(1 - \sigma)$ and the discounting rate

[11] Even if ρ is not unbounded, we can prove the uniqueness in the case of a monotonically increasing ρ.

$\rho = \tilde{\rho}(c^{1-\alpha}m^{\alpha})$ for $\sigma > 1$ and $0 < \alpha < 1$. Note that this felicity satisfies $u_c/\rho_c = u_m/\rho_m$. In this specification, the money demand function is

$$\varphi(c, R) = \frac{c}{R}\frac{\alpha}{1-\alpha}.$$

For any $c > 0$ and $R = f_k + \mu > 0$, positive money demand exists. As long as the BG condition is satisfied for any c, k, m, a unique steady state exists.

In the case of the TC model, we consider the transaction function $T(c, m) = T_0(m/c)^{-\eta}c$, where $T_0 > 0$ and $\eta > 0$. In this specification, the money demand function is

$$\varphi(c, R) = c\left(\frac{\eta T_0}{R}\right)^{1/(1+\eta)}.$$

For any $c > 0$ and $R = f_k + \mu > 0$, positive money demand exists. This transaction cost function satisfies $T_{mm}(1 + T_c) - T_{cm}T_m > 0$ for all $c > 0$ and $m > 0$, implying that gross consumption is increasing in net consumption for a given nominal interest rate or a given k. As long as the BG condition is satisfied for any c, k, m, a unique steady state with this specification exists.

5. Concluding Remarks

We have examined the uniqueness conditions for the steady state as well as the local stability properties in the MIUF and TC models with recursive utility, i.e., utility with an internally determined discount rate.

Assuming the existence of the steady state, we have found that a decreasing money demand function in the nominal interest rates and a monetary version of the BG condition, in which the increase in the discount rate dominates the increase in the marginal product in the steady-state equilibrium, ensure a negative determinant of the Jacobian matrix of the dynamic system, and that such conditions justify the comparative statics by Chen et al. (2008).

In the MIUF and TC models, when the money demand function is independent of consumption, the BG condition assures the local stability of the dynamics. When the money demand function is increasing in consumption, a sufficient condition for local stability is that impatience is marginally increasing for consumption in both models.

The MIUF model requires the money demand function to be independent of capital, the felicity function to be concave with respect to consumption and real balances, and the discount rate function to be concave (resp. convex) in the case of negative (resp. positive) felicity. When the money demand function is independent of capital, ρ_c and ρ_m have the same sign. The established conditions are consistent with Uzawa (1968), Fischer (1979), Calvo (1979), and Epstein and Hynes (1983).

In the TC model, unlike in the MIUF model, the range over which ρ_c is stable can be expressed explicitly using the production and transaction technology. This is when an increase in "net" consumption increases "gross" consumption for given nominal interest rate, and the transaction cost function is convex. Therefore, the range includes a constant time preference.

Although marginal impatience should be increasing in consumption as a sufficient condition for local stability, the possibility of decreasing marginal impatience exists. The analysis of the present study has demonstrated that DMI in consumption and real balances is more plausible for the condition in which the concavity of the production function becomes stronger, the money demand function becomes less elastic in both models, the curvatures of the felicity and the discount rate functions become stronger in the MIUF model, and the convexity of the transaction costs function becomes stronger in the TC model.

Furthermore, we have examined the conditions for the uniqueness of the steady state. If the BG condition is satisfied globally and the money demand function exists globally in both models and an increase in “net” consumption increases the “gross” consumption globally in the TC model, then the steady state is unique. An example providing an explicit money demand function can be provided for each model.

The analysis carried out in the present study has suggested a possibility for DMI. Accordingly, the dynamic systems of both models might be locally stable in the case of DMI in consumption and real balances, which is better supported by the empirical evidence. Conducting several quantitative exercises with a set of parameters reconciling empirical results (Becker and Mulligan, 1997, Ogaki and Atkinson, 1997, among others) may be an important future task.

Appendix

In this Appendix, we present a number of indicative mathematical manipulations. Appendix I deals with the MIUF model, and Appendix II deals with the TC model.

I The MIUF Model

As discussed in Subsection 2.1, the dynamic system in equilibrium is described by (4), (5), (6), (9), (10), $\phi = V(m+k)$, $\lambda = V'(m+k) = u_c - \rho_c V$, and the boundary conditions ($a_0 > 0$, $\lim_{t\to\infty} \lambda\,(m+k)\mathrm{e}^{-\Delta} = 0$ and $\lim_{t\to\infty} \phi\,\Delta\mathrm{e}^{-\Delta} = 0$, where $\Delta_t = -\int_0^t \rho\,(c_\tau, m_\tau)d\tau$).

Combining (4) and (5) by replacing λ yields $\pi = (u_m - \rho_m V)/(u_c - \rho_c V) - f_k$. Substituting this equation into (9) gives:

$$\dot{m}/m = \mu + f_k - \frac{u_m - \rho_m V}{u_c - \rho_c V}. \tag{37}$$

Taking the time derivatives in (4), and using $\lambda = V'(m+k) = u_c - \rho_c V$ and (6), we obtain:

$$(u_{cc} - \rho_{cc}V)\dot{c} + (u_{cm} - \rho_{cm}V)\dot{m} - \rho_c(u_c - \rho_c V)\dot{k} - \rho_c(u_c - \rho_c V)\dot{m} = (u_c - \rho_c V)(\rho - f_k).$$

Finally, equilibrium c, m, k is used to solve the first-order differential equations

$$\theta\dot{c}/c = f_k - \rho + \frac{u_{cm} - \rho_{cm}V}{u_c - \rho_c V}\dot{m} - \rho_c\dot{k} - \rho_c\dot{m}, \tag{38}$$

(10) and (37) being subject to the boundary conditions, where $\theta = -(u_{cc} - \rho_{cc}V)c/\ (u_c - \rho_c V)$ represents the intertemporal elasticity of substitution.

Linear approximation

The linear approximation around the steady state c^*, m^*, k^* gives:

$$\begin{bmatrix} \theta/c^* & \rho_c - K(c^*, m^*) & \rho_c \\ 0 & 1/m^* & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{c} \\ \dot{m} \\ \dot{k} \end{bmatrix}$$

$$= \begin{bmatrix} -\rho_c & -\rho_m & f_{kk} \\ -L_c & -L_m & -L_k + f_{kk} \\ -1 & 0 & f_k \end{bmatrix} \begin{bmatrix} c_t - c^* \\ m_t - m^* \\ k_t - k^* \end{bmatrix}, \tag{39}$$

where $K = (u_{cm} - \rho_{cm}V)/(u_c - \rho_c V)$, and $L_x = \partial\{(u_m - \rho_m V)/(u_c - \rho_c V)\}/\partial x$ for $x = k, c, m$ is represented as follows:

$$L_k = \frac{\rho_c u_m - \rho_m u_c}{u_c - \rho_c V}, \tag{40}$$

$$L_c = \frac{(u_c - \rho_c V)(u_{cm} - V\rho_{cm}) - (u_m - \rho_m V)(u_{cc} - V\rho_{cc})}{(u_c - \rho_c V)^2}, \tag{41}$$

$$L_m = \frac{(u_c - \rho_c V)(u_{mm} - V\rho_{mm}) - (u_m - \rho_m V)(u_{cm} - V\rho_{cm})}{(u_c - \rho_c V)^2} + L_k. \tag{42}$$

We denote the 3×3 matrix on the left-hand side of (39) by A_{MIUF} and the 3×3 matrix on the right-hand side of (39) by B_{MIUF}. Then,

$$\begin{bmatrix} \dot{c} \\ \dot{m} \\ \dot{k} \end{bmatrix} = J_{MIUF} \begin{bmatrix} c_t - c^* \\ m_t - m^* \\ k_t - k^* \end{bmatrix},$$

where J_{MIUF} is given by:

$$J_{MIUF} = A_{MIUF}^{-1} B_{MIUF}. \tag{43}$$

The determinant

The determinant is represented by:

$$\det(J_{MIUF}) = \frac{\det(B_{MIUF})}{\det(A_{MIUF})} = \frac{1}{\det(A_{MIUF})} \det \begin{bmatrix} -\rho_c & -\rho_m & f_{kk} \\ -L_c & -L_m & -L_k + f_{kk} \\ -1 & 0 & f_k \end{bmatrix}.$$

By performing a few algebraic operations, we can obtain (12) in Section 2.2.

The trace

Since,

$$A_{MIUF}^{-1} = \begin{bmatrix} \frac{c^*}{\theta} & -\frac{c^* m^* \rho_c}{\theta} + \frac{c^* m^*}{\theta} K & -\frac{c^* \rho_c}{\theta} \\ 0 & m^* & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

the trace is:

$$\begin{aligned}\mathrm{tr}(J_{MIUF}) &= \mathrm{tr}(A_{MIUF}^{-1}B_{MIUF})\\ &= \frac{c^*m^*}{\theta}\rho_c L_c - \frac{c^*m^*}{\theta}K(c^*,m^*)L_c - m^*L_m + f_k\\ &= \frac{c^*m^*}{\theta}\rho_c L_c - m^*L_k + f_k\\ &\quad + m^*\frac{(u_{cm}-\rho_{cm}V)^2-(u_{cc}-\rho_{cc}V)(u_{mm}-\rho_{mm}V)}{(u_c-\rho_c V)(u_{cc}-\rho_{cc}V)}\end{aligned}$$

Using $V = u/\rho$ at the steady state, we can obtain (14) in Section 2.2. Note that it follows from the second line that $\mathrm{tr}(J_{MIUF}) = -m^*L_m + f_k$ when $L_c = 0$.

Comparative statics

Under a steady state, in which $\dot{c} = \dot{m} = \dot{k} = 0$, the total differentiation of (38), (37), and (10) with respect to c, m, k, and μ yields:

$$B_{MIUF}\begin{bmatrix}\mathrm{d}c\\ \mathrm{d}m\\ \mathrm{d}k\end{bmatrix} + \begin{bmatrix}0\\ \mathrm{d}\mu\\ 0\end{bmatrix} = 0.$$

Therefore,

$$\begin{bmatrix}\mathrm{d}c/\mathrm{d}\mu\\ \mathrm{d}m/\mathrm{d}\mu\\ \mathrm{d}k/\mathrm{d}\mu\end{bmatrix} = B_{MIUF}^{-1}\begin{bmatrix}0\\ -1\\ 0\end{bmatrix} = \frac{1}{\det(B_{MIUF})}\begin{bmatrix}-\rho_m f_k\\ -(f_{kk}-\rho_c f_k)\\ -\rho_m\end{bmatrix}$$

Thus, we can obtain (11) in Section 2.2 because $\det(J_{MIUF}) = \det(B_{MIUF})/\det(A_{MIUF})$ and $\det(A_{MIUF}) = \theta/(c^*m^*)$.

II The TC Model

As discussed in Subsection 3.1, the dynamic system in equilibrium is described by (6), (9), (22), (23), (24), $\phi = V(m+k)$, $\lambda = V'(m+k) = u_c - \rho_c V$, and the boundary conditions ($a_0 > 0$, $\lim_{t\to\infty}\lambda(m+k)\mathrm{e}^{-\Delta} = 0$ and $\lim_{t\to\infty}\phi\,\Delta\mathrm{e}^{-\Delta} = 0$, where $\Delta_t = -\int_0^t \rho(c_\tau, m_\tau)d\tau$).
Combining (9) and (23) by replacing for π gives:

$$\dot{m}/m = \mu + f_k + T_m. \tag{44}$$

Taking the time derivatives in (22) and using $\lambda = (u_c - \rho_c V)/(1+T_c) = V'(m+k)$ and (6), we obtain:

$$\frac{(u_{cc}-\rho_{cc}V)\dot{c}}{1+T_c} - \frac{u_c-\rho_c V}{(1+T_c)^2}\{T_{cc}\dot{c} + T_{cm}\dot{m} + \rho_c(\dot{k}+\dot{m})\} = \frac{u_c-\rho_c V}{1+T_c}(\rho - f_k).$$

Finally, equilibrium c, m, k is used to solve the first-order differential equations:

$$\theta\dot{c}/c = f_k - \rho - \frac{T_{cm}}{1+T_c}\dot{m} - \frac{\rho_c}{1+T_c}\dot{m} - \frac{\rho_c}{1+T_c}\dot{k}, \tag{45}$$

(24) and (44) being subject to the boundary conditions, where $\theta = \tilde{\theta} + cT_{cc}/(1+T_c)$, and $\tilde{\theta} = -c(u_{cc}-\rho_{cc}V)/(u_c-\rho_c V)$ represents the intertemporal elasticity of substitution.

Linear approximation

The linear approximations around the steady state c^*, m^*, k^* yield:

$$\begin{bmatrix} \frac{\theta}{c^*} & \frac{\rho_c + T_{cm}}{1+T_c} & \frac{\rho_c}{1+T_c} \\ 0 & \frac{1}{m^*} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{c} \\ \dot{m} \\ \dot{k} \end{bmatrix} = \begin{bmatrix} -\rho_c & 0 & f_{kk} \\ T_{cm} & T_{mm} & f_{kk} \\ -(1+T_c) & -T_m & f_k \end{bmatrix} \begin{bmatrix} c_t - c^* \\ m_t - m^* \\ k_t - k^* \end{bmatrix}. \tag{46}$$

We denote the 3×3 matrix on the left-hand side of (46) by A_{TC} and the 3×3 matrix on the right-hand side of (46) by B_{TC}. That is,

$$\begin{bmatrix} \dot{c} \\ \dot{m} \\ \dot{k} \end{bmatrix} = J_{TC} \begin{bmatrix} c_t - c^* \\ m_t - m^* \\ k_t - k^* \end{bmatrix},$$

where J_{TC} is given by:

$$J_{TC} = A_{TC}^{-1} B_{TC}. \tag{47}$$

The determinant

The determinant is represented by:

$$\det(J_{TC}) = \frac{\det(B_{TC})}{\det(A_{TC})} = \frac{1}{\det(A_{TC})} \det \begin{bmatrix} -\rho_c & 0 & f_{kk} \\ T_{cm} & T_{mm} & f_{kk} \\ -(1+T_c) & -T_m & f_k \end{bmatrix}.$$

By performing a few algebraic operations, we can obtain (26) in Section 3.2.

The Trace

Since,

$$A_{TC}^{-1} = \begin{bmatrix} \frac{c^*}{\theta} & -\frac{c^* m^* (T_{cm} + \rho_c)}{\theta(1+T_c)} & -\frac{c^*}{\theta(1+T_c)} \rho_c \\ 0 & m^* & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

the trace is:

$$\begin{aligned} \mathrm{tr}(J_{TC}) = \mathrm{tr}(A_{TC}^{-1} B_{TC}) &= -\frac{c^* m^* (T_{cm} + \rho_c) T_{cm}}{\theta(1+T_c)} + m^* T_{mm} + f_k \\ &= \frac{(1+T_c)(u_{cc} - V\rho_{cc}) T_{mm}}{(1+T_c)(u_{cc} - V\rho_{cc}) - (u_c - V\rho_c) T_{cc}} m^* \\ &\quad - \frac{(u_c - V\rho_c)(T_{cc} T_{mm} - T_{cm}^2 - \rho_c T_{cm})}{(1+T_c)(u_{cc} - V\rho_{cc}) - (u_c - V\rho_c) T_{cc}} m^* + f_k. \end{aligned}$$

Using $V = u/\rho$ at the steady state, we can obtain (28) in Section 3.2. Note that it follows from the second line that $\mathrm{tr}(J_{MIUF}) = m^* T_{mm} + f_k$ when $T_{cm} = 0$.

Comparative statics

Under a steady state, in which $\dot{c} = \dot{m} = \dot{k} = 0$, the total differentiation of (45), (44), and (24) with respect to c, m, k, and μ yields:

$$B_{TC}\begin{bmatrix} \mathrm{d}c \\ \mathrm{d}m \\ \mathrm{d}k \end{bmatrix} + \begin{bmatrix} 0 \\ \mathrm{d}\mu \\ 0 \end{bmatrix} = 0.$$

Therefore,

$$\begin{bmatrix} \mathrm{d}c^*/\mathrm{d}\mu \\ \mathrm{d}m^*/\mathrm{d}\mu \\ \mathrm{d}k^*/\mathrm{d}\mu \end{bmatrix} = B_{TC}^{-1}\begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} = \frac{1}{\det(B_{TC})}\begin{bmatrix} T_m f_k \\ \rho_c f_k - (1+T_c) f_{kk} \\ -\rho_c T_m \end{bmatrix}$$

Thus, we can obtain (25) in Section 3.2 because $\det(J_{TC}) = \det(B_{TC})/\det(A_{TC})$ and $\det(A_{TC}) = \theta/(c^* m^*)$.

Acknowledgments

We would like to thank Nobusumi Sagara, Koichi Futagami, Junichi Itaya, Shinsuke Ikeda, and Toshiyuki Okuyama for helpful comments and suggestions. All the remaining errors are our own responsibility. The first author is supported by Grants-in-Aid for Scientific Research in Japan.

References

Becker, G.S., C.B. Mulligan (1997), The Endogenous Determination of Time Preference, *Quarterly Journal of Economics*, Vol. 112, No. 3, pp.729–758.

Brock, W.A., D. Gale (1969), Optimal Growth under Factor Augmenting Progress, *Journal of Economic Theory*, Vol. 1, pp.229–243.

Calvo, G.A. (1979), On Models of Money and Perfect Foresight, *International Economic Review*, Vol. 20, No. 1, pp.83–103.

Chang, F.R. (1994), Optimal Growth and Recursive Utility: Phase Diagram Analysis, *Journal of Optimization Theory and Applications* Vo. 80, 425–439.

Chen, B.L., M. Hsu, C.H. Lu (2008), Inflation and Growth: Impatience and a Qualitative Equivalence, *Journal of Money, Credit, and Banking*, Vol. 40, No. 3, pp.1309–1323.

Das, M. (2003), Optimal Growth with Decreasing Marginal Impatience, *Journal of Economic Dynamics and Control*, Vol. 27, pp.1881–1898.

Epstein, L.G., J.A. Hynes (1983), The Rate of Time Preference and Dynamic Economic Analysis, *Journal of Political Economy*, Vol. 91, No. 4, pp.611–635.

Fischer, S. (1979), Capital Accumulation on the Transition Path in a Monetary Optimizing Model, *Econometrica*, Vol. 47, No. 6, pp.1433–1439.

Hayakawa, H. (1995), The Complete Complementarity of Consumption and Real Balances and the Strong Superneutrality of Money, *Economics Letters*, Vol. 48, pp.91–97.

Hirose, K., S. Ikeda (2008), On Decreasing Marginal Impatience, *Japanese Economic Review* Vol. 59, No. 3, pp.259-274

Koopmans, T.C. (1960), Stationary Ordinal Utility and Impatience, *Econometrica*, Vol. 28, No. 3, pp.287–309.

Obstfeld, M. (1990), Intertemporal Dependence, Impatience, and Dynamics, *Journal of Monetary Economics*, Vol. 26, No. 1, pp.45–75.

Ogaki, M., A. Atkinson (1997), Rate of Time Preference, Intertemporal Elasticity of Substitution, and Level of Wealth, *Review of Economics and Statistics*, Vol. 79, No. 4, pp.564–572.

Saving, T.R. (1971), Transaction Costs and the Demand for Money, *American Economic Review*, Vol. 61, No. 3, pp.407–420.

Sidrauski, M. (1967), Inflation and Economic Growth, *Journal of Political Economy*, Vol. 75, No. 6, pp.796–810.

Tobin, J. (1965), Money and Economic Growth, *Econometrica*, Vol. 33, No. 4, pp.671–684.

Uzawa, H. (1968), Time Preference, Consumption, and Optimal Asset Holdings, in James, N.W. (ed) *Value, Capital, and Growth: Papers in Honor of Sir John Hicks*, Edinburgh: University of Edinburgh Press. pp.485–504.

Wang, P., C.K. Yip (1992), Alternative Approaches to Money and Growth, *Journal of Money, Credit, and Banking*, Vol. 24, No. 4, pp.553–562.

Zhang, J. (2000), Inflation and Growth: Pecuniary Transactions Costs and Qualitative Equivalence, *Journal of Money, Credit, and Banking*, Vol. 32, No. 1, pp.1–12.