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Representation of Preference Orderings with an Infinite Horizon: Time-additive Separable Utility in Continuous Time*

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Abstract

This paper proposes an axiomatic approach in a continuous-time framework for representing preference orderings with an infinite horizon in terms of time-additive separable (TAS) utility. To deal with divergent paths that emerge in endogenous growth models, we introduce several new axioms for preference orderings and exploit an integral representation of nonlinear functionals on L^p-spaces to obtain TAS utility functions with constant discount rates. Moreover, it is demonstrated that preference orderings given by recursive utility functionals are representable by means of TAS utility functions.

Keywords: Preference orderings, Infinite horizon, Time-additive separable utility, Integral representation on L^p-spaces, Recursive utility

JEL Classification: D91

1. Introduction

In the analysis of intertemporal resource allocations involving dynamic optimization, time-additive separable (TAS) utility functions have been widely used because of their simplicity and tractability. It was Koopmans (1972b) who first axiomatized, in a discrete time framework, preference orderings with an infinite horizon that are representable by TAS utility functions. The Koopmans formulation is intuitively transparent and general enough and, thus, it has served as a foundation for many applications in intertemporal decision making.

The approach Koopmans developed is as follows. Preference orderings with an infinite horizon are truncated to those with a finite horizon to represent those in terms of finitely additive separable utility functions, employing the axiomatization of Debreu (1960), Gorman (1968) and Koopmans (1972a) on the separability of utility functions; the finite sum of the instantaneous utility functions is then extended to the countably infinite sum of those by a certain kind of limiting argument. The advantage of this truncation method lies in the fact that constant discount rates are derivable in the existence argument of TAS utility functions.

The choice space Koopmans adopted is a subset of l^∞ (the space of bounded sequences) with the sup norm. It precludes divergent paths that emerge in endogenous growth models. To

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overcome this difficulty, Dolmas (1995) generalized Koopmans's result to encompass divergent paths that "grow no faster than a fixed reference path," and Bleichrodt et al. (2008) and Harvey (1986) obtained a TAS representation admitting more general unbounded paths under the finite-dimensional continuity of preference orderings, which is a weaker requirement than the sup norm continuity. These works exploit the same truncation method as that of Koopmans (1972b).

It is quite natural to pursue the TAS representability in continuous time under the same axioms posed by Koopmans (1972b) in discrete time. In the literature, however, there is no "exact" counterpart in continuous time. One of the reasons is a difficulty in applying the truncation method of Koopmans and others because the choice space L^∞ (the space of essentially bounded functions) in continuous time is infinite dimensional, even if time horizons are fixed to be finite, and it lacks topological separability under the sup norm, which prevents one from employing the utility representation theorem of Debreu (1960) and Gorman (1968).[1]

The purpose of this paper is to present an axiomatic approach in a continuous-time framework for representing preference orderings with an infinite horizon in terms of integral functionals. Some axioms imposed here are similar to those of Koopmans (1972b), but some are completely different. While we employ the variants of axioms that are akin to continuity, nontriviality and independence in decision theory under uncertainty along the lines of Savage (1972) and others, which are also imposed by Koopmans and adherents (Bleichorodt et. al. 2008, Dolmas 1995, Harvey 1986 and Koopmans 1972b) in intertemporal preferences without uncertainty, the technique presented in this paper greatly differs from that in the previous works.

To deal with divergent paths that emerge in endogenous growth models and nonconvex growth problems with increasing returns, incorporating discount factors explicitly, we exploit integral representations of nonlinear functionals on L^p-spaces investigated by Buttazzo and Dal Maso (1983) and applied to nonconvex variational problems by Sagara (2007). The choice space under consideration is an admissible subset of an L^p-space. The reason for adopting an L^p-space lies in its topological separability under the L^p-norm for the sake of the existence of utility functions. By employing the Riesz representation theorem, Weibull (1985) obtained in continuous time a TAS representation of linear preferences on an L^1-space in terms of a linear functional. Our technique is based on a generalized "nonlinear version" of the Riesz representation theorem.[2]

Because we permit a freedom for the choice of measures (including the Lebesgue measure) on the time interval, divergent paths can be treated in a simple manner, and, hence, the L^∞-space with respect to the Lebesgue measure can be naturally included in an L^p-space with respect to a given measure. The space under investigation is called a *weighted L^p-space*. A space of this type was first introduced to economic growth theory by Chichilnisky (1977), who employed a *weighted Sobolev space*, to deal with unbounded paths and nonconvexities in an infinite horizon.

An alternative approach to the TAS representation in continuous time was explored by Epstein (1987), who hypothesized a flexible rate of time preference to obtain a representation in terms of TAS and recursive utility functions by specifying the properties of a "generating function," by which TAS and recursive utility functions are obtained as a solution of an ordinary differential equation (ODE). The technique is based on a continuous time analog of an "aggregator function" introduced by Koopmans (1960) and then elaborated by Lucas and Stokey (1984) in discrete time to formulate intertemporal utility with an infinite horizon. Epstein (1987) requires, however, the strong assumption that preference orderings are representable by continuously differentiable utility functions, which is absent in Koopmans (1972b).

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[1] Note that Koopmans (and Debreu 1960) already recognized this point. See Koopmans (1960, p.291, Footnote 5) and Koopmans (1972a, p.84, Footnote 7).

[2] Fonesca and Leoni (2007) is a useful textbook on integral representations of nonlinear functionals on L^p-spaces.

The paper proceeds as follows. We show in Section 2 that if a preference ordering on an admissible subset of an L^p-space satisfies *strong continuity*, *disjoint independence*, *local sensitivity*, *local substitutability* and *disjoint additivity*, then there exists a continuous TAS representation for the preference ordering such that it is an integral functional with a continuous integrand (instantaneous utility function) satisfying a certain kind of boundedness (Theorem 2.1). Consequently, a continuous TAS utility function with a constant discount rate can be obtained. Moreover, if the preference ordering satisfies the continuity with respect to the weak topology of the L^p-space, then the integrand becomes a concave integrand even without assuming the convexity of the preference ordering (Theorem 2.3).

It is well known that the use of TAS utility has been criticized by various authors. For example, Lucas and Stokey (1984, p.169) manifested their opinion that "time-additivity is neither a desirable nor an analytically necessary property to impose on preferences." In Section 3, it is demonstrated that preference orderings given by recursive utility functionals along the lines of Epstein (1987a, 1987b), Epstein and Hynes (1983) and Uzawa (1968) are representable by means of TAS utility functions (Theorem 3.1), which is a variant of the result by Sagara (2007). This implies that our result serves to advocate the use of TAS utility.

2. TAS Representation on L^p-Spaces

2.1. Preliminaries

Let $(\mathcal{R}_+,\mathcal{L})$ be a Lebesgue measurable space, in which $\mathcal{R}_+=[0,\infty)$ is the time interval with an infinite horizon and $\mathcal{L}$ is the σ-field of Lebesgue measurable subsets of $\mathcal{R}_+$. A measure μ of the Lebesgue measurable space $(\mathcal{R}_+,\mathcal{L})$ is a *Borel measure* if $\mu(K)<\infty$ for every compact subset K of $\mathcal{R}_+$. It is said to be *complete* if an arbitrary subset of a μ-null set belongs to $\mathcal{L}$. We assume in the sequel that μ is a complete Borel measure that is absolutely continuous with respect to the Lebesgue measure. We use the phrase "a.e. $t\in\mathcal{R}_+$" to mean that the underlying measure is the Lebesgue measure; otherwise we employ "μ-a.e. $t\in\mathcal{R}_+$".

Let $(\mathcal{R}^n,\mathcal{B}^n)$ be a Borel measurable space with $\mathcal{B}^n$ the σ-field of Borel subsets of $\mathcal{R}^n$. For every real number $1\le p<\infty$, let $L^p(\mathcal{R}_+,\mu;\mathcal{R}^n)$ be the set of Lebesgue measurable functions x from $\mathcal{R}_+$ to $\mathcal{R}^n$ such that $\int_0^\infty |x(t)|^p d\mu(t)<\infty$, endowed with the L^p-norm, where $|\cdot|$ is the Euclidean norm of $\mathcal{R}^n$. Since $\mathcal{L}$ is countably generated, $L^p(\mathcal{R}_+,\mu;\mathcal{R}^n)$ is a separable Banach space (see Fonesca and Leoni 2007, Theorem 2.16). We use $L^p(\mathcal{R}_+,\mu)$ when $n=1$ and simply denote $L^p(\mathcal{R}_+;\mathcal{R}^n)$ when μ is the Lebesgue measure of $(\mathcal{R}_+,\mathcal{L})$. By $L^1_{\mathrm{loc}}(\mathcal{R}_+,\mu)$, we denote the space of locally integrable functions on $\mathcal{R}_+$ with respect to the measure μ.

Let χ_A be the characteristic function of $A\in\mathcal{L}$; that is, $\chi_A(t)=1$ if $t\in A$ and $\chi_A(t)=0$ otherwise. If x is a *trajectory* in $L^p(\mathcal{R}_+,\mu;\mathcal{R}^n)$, then $x\chi_A$ denotes a trajectory in $L^p(\mathcal{R}_+,\mu;\mathcal{R}^n)$ taking its values $x(t)$ for $t\in A$ and zero on $\mathcal{R}_+\setminus A$. Thus, if x and y are trajectories in $L^p(\mathcal{R}_+,\mu;\mathcal{R}^n)$ and $A\cap B=\emptyset$, then $x\chi_A+y\chi_B$ is a "patched" trajectory in $L^p(\mathcal{R}_+,\mu;\mathcal{R}^n)$ taking its values $x(t)$ for $t\in A$ and $y(t)$ for $t\in B$ and vanishing on $\mathcal{R}_+\setminus(A\cup B)$.

2.2. Axioms for Preference Orderings

In the sequel, a choice space $\mathcal{X}$ is a subset of $L^p(\mathcal{R}_+,\mu;\mathcal{R}^n)$ endowed with the relative (norm or weak) topology from $L^p(\mathcal{R}_+,\mu;\mathcal{R}^n)$. The range of a choice space $\mathcal{X}$ is a subset of $\mathcal{R}^n$ given by $X=\{x(\mathcal{R}_+)\mid x\in\mathcal{X}\}$, which is the set of possible values for $\mathcal{X}$. Given $v\in X$ and $A\in\mathcal{L}$ with $0<\mu(A)<\infty$, we say that a trajectory $v\chi_A$ in $L^p(\mathcal{R}_+,\mu;\mathcal{R}^n)$ is *locally constant*.

Definition 2.1. A subset $\mathcal{X}$ of $L^p(\mathcal{R}_+,\mu;\mathcal{R}^n)$ is *admissible* if the following conditions are satisfied: (i) $0 \in \mathcal{X}$; (ii) $x, y \in \mathcal{X}$ and $A \cap B = \emptyset$ imply $x\chi_A + y\chi_B \in \mathcal{X}$.

Whenever $\mathcal{X}$ is an admissible set, $x \in \mathcal{X}$ if and only if $x\chi_A \in \mathcal{X}$ for every $A \in \mathcal{L}$. Hence, $\mathcal{X}_A = \{x\chi_A | x \in \mathcal{X}\}$ is contained in $\mathcal{X}$. An exemplary instance of an admissible set is a positive cone of $L^p(\mathcal{R}_+,\mu;\mathcal{R}^n)$ given by:

$$\mathcal{X} = \{x \in L^p(\mathcal{R}_+,\mu;\mathcal{R}^n) | x \geq 0\}. \tag{2.1}$$

A preference ordering $\succsim$ is a complete transitive binary relation defined on the admissible set $\mathcal{X}$. A *utility function* for a pair $(\mathcal{X},\succsim)$ is a real-valued function I on $\mathcal{X}$ such that $x \succsim y$ if and only if $I(x) \geq I(y)$. We say that $(\mathcal{X},\succsim)$ admits a *TAS representation* if it has a utility function I of the integral functional form: $I(x) = \int_0^\infty f(t, x(t))dt$ for $x \in \mathcal{X}$ with an $\mathcal{L} \times \mathcal{B}^n$-measurable function $f: \mathcal{R}_+ \times \mathcal{R}^n \to \mathcal{R}$; I is called a *TAS utility function*.

We introduce the following axioms on $(\mathcal{X},\succsim)$.

Axiom 2.1 (**Strong continuity**). For every $x \in \mathcal{X}$, the upper contour set $\{y \in \mathcal{X} | y \succsim x\}$ and the lower contour set $\{y \in \mathcal{X} | x \succsim y\}$ are L^p-norm closed in $\mathcal{X}$.

Axiom 2.2 (**Disjoint independence**). For every $A, B \in \mathcal{L}$ with $A \cap B = \emptyset$, $x\chi_A \succsim y\chi_A$ implies $x\chi_A + z\chi_B \succsim y\chi_A + z\chi_B$ for every $z \in \mathcal{X}$.

Axiom 2.3 (**Local sensitivity**). For every $A \in \mathcal{L}$ with $\mu(A) > 0$, there exist x and y in $\mathcal{X}$ such that $x\chi_A \succ y\chi_A$.

Axiom 2.4 (**Local substitutability**). For every $x \in \mathcal{X}$ and $A \in \mathcal{L}$ with $\mu(A) > 0$, there exists some $y \in \mathcal{X}$ such that $x \sim y\chi_A$.

Axiom 2.5 (**Disjoint additivity**). For every $x, y \in \mathcal{X}$ and $A, B, E, F \in \mathcal{L}$ satisfying $A \cap B = \emptyset$ and $E \cap F = \emptyset$, $x\chi_A \sim y\chi_E$ and $x\chi_B \sim y\chi_F$ imply that $x\chi_A + x\chi_B \sim y\chi_E + y\chi_F$.

Remarks on the Axioms

Strictly speaking, it is impossible to compare directly the axioms imposed above with those of Koopmans (1972b) because the relevant spaces are mathematically different (l^∞ with the sup norm versus L^p-space with the L^p-norm). However, it might be informative to see the similarity in the axioms and where we depart from the Koopmans axioms.

Strong continuity (Axiom 2.1) is a standard axiom in utility theory and it is imposed by Koopmans.

Disjoint independence (Axiom 2.2) implies that for every $A \in \mathcal{L}$, the preference ordering on $\mathcal{X}$ induces a preference ordering on $\mathcal{X}_A$ by restricting $\succsim$ to $\mathcal{X}_A + z\chi_{\mathcal{R}_+ \setminus A}$ for an arbitrarily fixed trajectory $z \in \mathcal{X}$. It is a counterpart of the Koopmans axiom called *complete independence* and obviously a weaker axiom than:

Axiom 2.2* (**Independence**). For every $x, y, z \in \mathcal{X}$, $x \succsim y$ implies $x + z \succsim y + z$ whenever $x + z, y + z \in \mathcal{X}$.

Axiom 2.2* (independence) is imposed by Weibull (1985) in continuous time.

Local sensitivity (Axiom 2.3) rules out the situation where the induced preference ordering on $\mathcal{X}_A$ with $\mu(A) > 0$ is "degenerate" in that every trajectory in $\mathcal{X}_A$ is indifferent. It corresponds to the Koopmans axiom of *sensitivity*.

Local substitutability (Axiom 2.4) might seem a somewhat strong requirement when maximal elements in $\mathcal{X}$ with respect to $\succsim$ exist, because it necessarily implies the existence of multiple maximal elements. In particular, if $\mathcal{X}$ is convex, then it excludes the strict convexity of $(\mathcal{X}, \succsim)$, which guarantees a unique maximal element. Note that, together with strong continuity (Axiom 2.1), there exists a maximal element in $\mathcal{X}$ provided that $\mathcal{X}$ is compact. In most economic applications, however, it is a common situation that a choice space in itself is not compact, but its feasible subset is compact. Because we do not assume the compactness of $\mathcal{X}$, local substitutability (Axiom 2.4) is not a strong restriction whenever maximal elements are nonexistent in $\mathcal{X}$, but "optimal" elements are existent in the feasible subset of $\mathcal{X}$. For instance, linear preference orderings in subsection 2.3 satisfy local substitutability (Axiom 2.4). This is not imposed by Koopmans (1972b).

Disjoint additivity (Axiom 2.5), which Koopmans did not postulate, essentially implies disjoint independence (Axiom 2.2). More precisely, disjoint additivity (Axiom 2.5) implies the following weaker form of disjoint independence:

Axiom 2.2.** For every $A, B \in \mathcal{L}$ with $A \cap B = \emptyset$, $x\chi_A \sim y\chi_A$ implies $x\chi_A + z\chi_B \sim y\chi_A + z\chi_B$ for every $z \in \mathcal{X}$.

To demonstrate this claim, let $x, y, z \in \mathcal{X}$ and $A \cap B = \emptyset$. Suppose that $x\chi_A \sim y\chi_A$. Define $v = x\chi_A + z\chi_B$ and $w = y\chi_A + z\chi_B$. Since $v\chi_A = x\chi_A$, $w\chi_A = y\chi_A$ and $v\chi_B = w\chi_B = z\chi_B$, by construction, we have $v\chi_A \sim w\chi_A$ and $v\chi_B \sim w\chi_B$. Disjoint additivity (Axiom 2.5) implies $v\chi_A + v\chi_B \sim w\chi_A + w\chi_B$, which is equivalent to $x\chi_A + z\chi_B \sim y\chi_A + z\chi_B$, from which Axiom 2.2** follows.

A strengthened version of disjoint additivity (Axiom 2.5) is:

Axiom 2.5*. For every $x, y \in \mathcal{X}$ and $A, B, E, F \in \mathcal{L}$ satisfying $A \cap B = \emptyset$ and $E \cap F = \emptyset$, $x\chi_A \succsim y\chi_E$ and $x\chi_B \succsim y\chi_F$ imply $x\chi_A + x\chi_B \succsim y\chi_E + y\chi_F$.

If Axiom 2.5* is imposed instead of disjoint additivity (Axiom 2.5), it implies each variant of the independence axiom: Axiom 2.2, Axioms 2.2* and 2.2**.

Koopmans assumed the existence of maximal and minimal elements in $\mathcal{X}$ under the name of *extreme programs*, a superfluous axiom for our purpose. Other axioms that Koopmans required but are absent from this paper are *monotonicity* and *stationarity*.

Because the monotonicity axiom Koopmans introduced is quite different from the standard monotonicity axiom in decision theory, we call it *K-monotonicity*, as in Dolmas (1995)

Axiom 2.6 (**K-monotonicity**). $x\chi_{[0,t)} + y\chi_{[t,\infty)} \succsim x\chi_{[0,s)} + y\chi_{[s,\infty)}$ for every $s, t \in \mathcal{R}_+$ with $0 < s \leq t$ implies $x \succsim y$.

To introduce the stationarity axiom in our framework, we need the following condition for an admissible set. For every $x \in \mathcal{X}$ and $s \in \mathcal{R}_+$, define the trajectory x_s by $x_s(t) = x(s+t)$ for $t \in \mathcal{R}_+$. An admissible set $\mathcal{X}$ is *time invariant* if $x \in \mathcal{X}$ implies $x_s \in \mathcal{X}$ for every $s \in \mathcal{R}_+$.

Axiom 2.7 (**Stationarity**). Let $\mathcal{X}$ be a time-invariant admissible set. There exists $z \in \mathcal{X}$ such that for every $s \in \mathcal{R}_+$: $x_s \succsim y_s$ if and only if $z\chi_{[0,s)} + x\chi_{[s,\infty)} \succsim z\chi_{[0,s)} + y\chi_{[s,\infty)}$.

It is clear that under disjoint independence (Axiom 2.2), stationarity (Axiom 2.7) can be strengthened to:

Axiom 2.7*. For every $s \in \mathcal{R}_+$, $x_s \succsim y_s$ if and only if $z\chi_{[0,s)} + x\chi_{[s,\infty)} \succsim z\chi_{[0,s)} + y\chi_{[s,\infty)}$ for every $z \in \mathcal{X}$.

Instead of K-monotonicity (Axiom 2.6) and stationarity (Axiom 2.7), we require an alternative axiom in subsection 4, *locally constant indifference* (Axiom 2.8), to derive a time-independent integrand in the TAS representation.

2.3. TAS Representation I

Theorem 2.1. *Suppose that the admissible set* $\mathcal{X}$ *is closed and connected in the* L^p*-norm topology. If* $(\mathcal{X}, \succsim)$ *satisfies Axioms 2.1 to 2.5, then it admits a continuous TAS representation* $I(x) = \int_0^\infty f(t, x(t))\rho(t)dt$ *for* $x \in \mathcal{X}$ *with the following properties:*

(i) $f(t,\cdot)$ *is continuous on* $\mathcal{R}^n$ μ*-a.e.* $t \in \mathcal{R}_+$ *and* $f(\cdot, v)$ *is Lebesgue measurable on* $\mathcal{R}_+$ *for every* $v \in \mathcal{R}^n$.[3]

(ii) *There exist some* $\alpha \in L^1(\mathcal{R}_+, \mu)$ *and* $\beta \ge 0$ *such that*

$$|f(t,v)| \le \alpha(t) + \beta |v|^p \quad \mu\text{-a.e. } t \in \mathcal{R}_+ \text{ for every } v \in \mathcal{R}^n.$$[4]

Here, $f(t,0) = 0$ μ*-a.e.* $t \in \mathcal{R}_+$ *and* ρ *is uniquely determined and independent of the representation of I. Moreover, if* g *is another integrand with* $I = \int_0^\infty g\,\rho dt$ *that satisfies the condition (ii) with* $g(t,0) = 0$ μ*-a.e.* $t \in \mathcal{R}_+$*, then* $f(t,\cdot) = g(t,\cdot)$ μ*-a.e.* $t \in \mathcal{R}_+$*. Furthermore, if* μ *is a finite measure, then* ρ *is Lebesgue integrable and*

$$\liminf_{t\to\infty} \rho(t) = 0. \tag{2.2}$$

The theorem also reveals that the function ρ plays the role of a discount factor and the finiteness of μ implicitly entails eventually discounting the future. In particular, the condition (2.2) of the theorem implies that if μ has a continuous Radon–Nikodym derivative ρ, then $\lim_{t\to\infty} \rho(t) = 0$. For example, this is the case for the finite, complete, Borel measure μ_ρ given by

$$\mu_\rho(A) = \int_A \rho(t)dt \quad \text{for } A \in \mathcal{L} \tag{2.3}$$

with a positive, continuous, Lebesgue integrable function ρ.

Taking into account the fact that utility functions are unique up to a strictly increasing transformation, the theorem asserts that if J is another TAS utility function for $(\mathcal{X}, \succsim)$ with an integrand g satisfying $J(x) = \int_0^\infty g(t, x(t))\rho(t)dt$ for $x \in \mathcal{X}$ with $g(t,\cdot) = 0$ μ-a.e. $t \in \mathcal{R}_+$ (note that the unique weight function ρ is independent of the particular TAS representations I and J), there exists a strictly increasing continuous function $\Psi: \mathcal{R} \to \mathcal{R}$ such that $J = \Psi \circ I$ and $\Psi(0) = 0$.

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[3] A function $f: \mathcal{R}_+ \times \mathcal{R}^n \to \mathcal{R}$ satisfying the condition (i) is called a *Carathéodory integrand.*

[4] The condition (ii) is called the *growth condition.*

Corollary 2.1. *Suppose that $\mathcal{X}$ contains every locally constant trajectory and μ is a finite measure. Then J is a positive affine transformation of I if and only if g is a positive affine transformation of f on $\mathcal{R}_+ \times X$.*

The corollary states that the integrand g is not a positive affine transformation of f if and only if Ψ is not positively affine. Therefore, in the TAS representation under Axioms 2.1 to 2.5, the integrand f is not necessarily unique up to a positive affine transformation.

Finitely Additive Representation

Let $\{\Omega_1, \dots, \Omega_m\}$ be a partition of $\mathcal{R}_+$ with $m \geq 3$ such that each Ω_k has a positive measure. Define $\mathcal{X}_k = \mathcal{X}_{\Omega_k}$ for each $k = 1, \dots, m$. Since each trajectory $x \in \mathcal{X}$ is identified with a trajectory $(x\chi_{\Omega_1}, \dots, x\chi_{\Omega_m})$ in the product space $\prod_{k=1}^{m} \mathcal{X}_k$ and each element $(x_1, \dots, x_m) \in \prod_{k=1}^{m} \mathcal{X}_k$ is identified with its algebraic sum $\sum_{k=1}^{m} x_k \in \mathcal{X}$, it follows that $\mathcal{X} = \prod_{k=1}^{m} \mathcal{X}_k = \sum_{k=1}^{m} \mathcal{X}_k$, where $\sum_{k=1}^{m} \mathcal{X}_k$ is the algebraic sum of $\mathcal{X}_1, \dots, \mathcal{X}_m$.

Lemma 2.1.[5][6] *Suppose that the admissible set $\mathcal{X}$ is connected in the L^p-norm topology. If $(\mathcal{X}, \succsim)$ satisfies Axioms 2.1 to 2.3, then there exists a continuous function I_k on $\mathcal{X}_k$ for $k = 1, \dots, m$ such that*

$$x \succsim y \Leftrightarrow \sum_{k=1}^{m} I_k(x_k) \geq \sum_{k=1}^{m} I_k(y_k).$$

Proof. Let $K = \{1, \dots, m\}$ and L be an arbitrary subset of K. Since $(\mathcal{X}, \succsim)$ satisfies disjoint independence (Axiom 2.2), it induces on the product space $\prod_{k \in L} \mathcal{X}_k$ a preference ordering $\succsim_L$ such that for an arbitrarily fixed trajectory $z \in \mathcal{X}$:

$$(x_k)_{k \in L} \succsim_L (y_k)_{k \in L} \overset{\text{def}}{\Leftrightarrow} [(x_k)_{k \in L}, (z_k)_{k \in K \setminus L}] \succsim [(x_k)_{k \in K}, (z_k)_{k \in K \setminus L}].$$

We denote $\succsim_{\{k\}}$ by $\succsim_k$. Thus, for every subset L of K, the preference ordering $\succsim_L$ on $\prod_{k \in L} \mathcal{X}_k$ is independent of any $(z_k)_{k \in K \setminus L} \in \prod_{k \in K \setminus L} \mathcal{X}_k$. By local sensitivity (Axiom 2.3), there exist x_k and y_k in $\mathcal{X}_k$ such that $x_k \succ_k y_k$ for each $k \in K$.

Let pr_k be the projection from $\mathcal{X}$ onto $\mathcal{X}_k$. Then $\mathcal{X}_k$ is a connected set as the image of the connected set $\mathcal{X}$ by the continuous mapping pr_k. Since a metric space is separable if and only if it has a countable base of open sets, $\mathcal{X}$ and $\mathcal{X}_k$ are separable metric spaces as a subset of the separable Banach space $L^p(\mathcal{R}_+, \mu; \mathcal{R}^n)$. Since each $\mathcal{X}_k$ is connected, we can apply the classical theorem of Debreu– Gorman (see Debreu 1960 and Gorman 1968) to obtain an additive separable utility function for $(\mathcal{X}, \succsim)$. $\square$

By virtue of Lemma 2.1, there exists a continuous utility function I for $(\mathcal{X}, \succsim)$ with the form $I(x) = \sum_{k \in K} I_k(x_k)$. Without loss of generality, one may assume that $I_k(0) = 0$ for each $k \in K$.

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[5] Since the finitely additive representation on $(\mathcal{X}, \succsim)$ depends on the choice of a partition of $\mathcal{R}_+$, it might not be unique up to a positive affine transformation. However, this does not cause a problem in obtaining the uniqueness of f and ρ in Theorem 2.1.

[6] The requirement $m \geq 3$ is not removable for a finitely additive separable representation. Koopmans (1972a) gave a counter example such that for $m = 2$, there exists a preference ordering on a connected separable topological space satisfying (in our terminology) strong continuity (Axiom 2.1), disjoint independence (Axiom 2.2) and local sensitivity (Axiom 2.3) such that it cannot be represented by an additive separable utility function.

Lemma 2.2. *Suppose that the admissible set $\mathcal{X}$ is connected in the L^p-norm topology. If $(\mathcal{X}, \succsim)$ satisfies Axioms 2.1 to 2.5, then I is disjointly additive on $\mathcal{X}$, that is, $x, y \in \mathcal{X}$ and $A \cap B = \emptyset$ imply $I(x\chi_A + y\chi_B) = I(x\chi_A) + I(y\chi_B)$.*

Proof. Take any $x \in \mathcal{X}$ and $A, B \in \mathcal{L}$ with $A \cap B = \emptyset$. Let $E, F \in \mathcal{L}$ be of positive measure such that $E \subset \bigcup_{k\in K_1} \Omega_k$ and $F \subset \bigcup_{k\in K_2} \Omega_k$ for some partition $\{K_1, K_2\}$ of K. Then E and F are disjoint. By local substitutability (Axiom 2.4), there exist u and v in $\mathcal{X}$ such that $x\chi_A \sim u\chi_E$ and $x\chi_B \sim v\chi_F$. Define $y = u\chi_E + v\chi_F$. Since $\mathcal{X}$ is admissible, we have $y \in \mathcal{X}$. Note that $y\chi_E = u\chi_E$ and $y\chi_F = v\chi_F$. Thus, we have $x\chi_A \sim y\chi_E$ and $x\chi_B \sim y\chi_F$. By disjoint additivity (Axiom 2.5), we have $x\chi_A + x\chi_B \sim y\chi_E + y\chi_F$. Define $E_k = E \cap \Omega_k$ and $F_k = F \cap \Omega_k$ for each $k \in K$. Then E and F are decomposed into n-tuples of pairwise disjoints sets $\{E_k\}_{k\in K}$ and $\{F_k\}_{k\in K}$, respectively, with $E_k = \emptyset$ for $k \in K_2$ and $F_j = \emptyset$ for $j \in K_1$. Thus, we have $y\chi_E = (y\chi_{E_1}, \dots, y\chi_{E_m}) \in \prod_{k\in K} \mathcal{X}_k$ with $y\chi_{E_k} = 0$ for $k \in K_2$ and $y\chi_F = \left(y\chi_{F_1}, \dots, y\chi_{F_m}\right) \in \prod_{k\in K} \mathcal{X}_k$ and $y\chi_{F_j} = 0$ for $j \in K_1$. Therefore, $I(x\chi_A) = I(y\chi_E) = \sum_{j\in K_1} I_j\left(y\chi_{E_j}\right)$, $I(x\chi_B) = I(y\chi_F) = \sum_{k\in K_2} I_k\left(y\chi_{F_k}\right)$ and $I(x\chi_A + y\chi_B) = I(y\chi_E + y\chi_F) = \sum_{j\in K_1} I_j\,(y\chi_{E_j}) + \sum_{k\in K_2} I_k\,(y\chi_{F_k})$, and, hence, $I(x\chi_A + x\chi_B) = I(x\chi_A) + I(x\chi_B)$.

From this observation, we can derive the disjoint additivity of I. To demonstrate this, let $x, y \in \mathcal{X}$ and $A \cap B = \emptyset$. Define $z = x\chi_A + y\chi_B$. We then have $z \in \mathcal{X}$ by the admissibility of $\mathcal{X}$, and $z\chi_A + z\chi_B = x\chi_A + y\chi_B$, by construction. Thus, $I(x\chi_A + y\chi_B) = I(z\chi_A + z\chi_B) = I(z\chi_A) + I(z\chi_B) = I(x\chi_A) + I(y\chi_B)$. □

Proof of Theorem 2.1

Define the functional $\Phi: L^p(\mathcal{R}_+, \mu; \mathcal{R}^n) \times \mathcal{L} \to \mathcal{R} \cup \{-\infty\}$ by

$$\Phi(x, A) = \begin{cases} I(x\chi_A) & \text{if } x\chi_A \in \mathcal{X}, \\ -\infty & \text{otherwise.} \end{cases}$$

By Lemmas 2.1 and 2.2, Φ satisfies the following properties:

- $\Phi(\cdot, \mathcal{R}_+)$ is upper semicontinuous on $L^p(\mathcal{R}_+, \mu; \mathcal{R}^n)$ with respect to the L^p-norm topology;
- Φ is *finitely additive* on $\mathcal{L}$, that is, $A, B \in \mathcal{L}$ and $A \cap B = \emptyset$ imply $\Phi(x, A \cup B) = \Phi(x, A) + \Phi(x, B)$ for every $x \in L^p(\mathcal{R}_+, \mu; \mathcal{R}^n)$;
- Φ is *local* on $\mathcal{L}$, that is, $x, y \in L^p(\mathcal{R}_+, \mu; \mathcal{R}^n)$ and $x\chi_A = y\chi_A$ imply $\Phi(x, A) = \Phi(y, A)$;
- $\Phi(0, A) = 0$ for every $A \in \mathcal{L}$.

Here, we show only the finite additivity of Φ because the other properties are evident. Suppose that A and B are disjoint. If $x\chi_{A\cup B} \in \mathcal{X}$, then the result follows immediately from Lemma 2.2. If $x\chi_{A\cup B} \notin \mathcal{X}$, then $x\chi_A$ and $x\chi_B$ by the admissibility of $\mathcal{X}$, and, hence $x\chi_A \notin \mathcal{X}$ or $x\chi_B \notin \mathcal{X}$; for otherwise, $x\chi_A \in \mathcal{X}$ and $x\chi_B \in \mathcal{X}$ yield $x\chi_{A\cup B} = x\chi_A + x\chi_B \in \mathcal{X}$ by the admissibility of $\mathcal{X}$, which is a contradiction. Therefore, $\Phi(x, A) + \Phi(x, B) = -\infty = \Phi(x, A \cup B)$.

Note also that μ is nonatomic because of the nonatomicity of the Lebesgue measure with respect to which μ is absolutely continuous and that the Borel measure μ is σ-finite (see Halmos 1950, Sec.52(1)). Thus, $(\mathcal{R}_+, \mathcal{L}, \mu)$ is a σ-finite complete nonatomic measure space. Then, by the representation theorem of Buttazzo and Dal Maso (1983), there exists a normal integrand[7] $(-f)\colon \mathcal{R}_+ \times \mathcal{R}^n \to \mathcal{R} \cup \{+\infty\}$ with the following properties:

(a) There exist some $\alpha \in L^1(\mathcal{R}_+, \mu)$ and $\beta \geq 0$ such that $f(t,v) \leq \alpha(t) + \beta|v|^p$ μ-a.e. $t \in \mathcal{R}_+$ for every $v \in \mathcal{R}^n$.

(b) $\Phi(x, A) = \int_A f\,(t, x(t))d\mu(t)$ for every $x \in L^p(\mathcal{R}_+, \mu; \mathcal{R}^n)$ and $A \in \mathcal{L}$.

It is evident from (a) that f satisfies the condition (ii). From the construction of Φ, we have $I(x) = \int_0^\infty f\,(t, x(t))\rho(t)dt$ for $x \in \mathcal{X}$, where ρ is the Radon–Nikodym derivative of μ with respect to the Lebesgue measure. Because $\int_A f\,(t, 0)\rho(t)dt = 0$ for every $A \in \mathcal{F}$ by (b), we have $f(t, 0) = 0$ μ-a.e. $t \in \mathcal{R}_+$. Let g be an integrand of I that satisfies the condition (ii) with $g(t, 0) = 0$ μ-a.e. $t \in \mathcal{R}_+$. We then have

$$\Phi(x, A) = \int_A f\,(t, x(t))\rho(t)dt = \int_0^\infty f\,(t, x\chi_A(t))\rho(t)dt$$
$$= \int_0^\infty g\,(t, x\chi_A(t))\rho(t)dt = \int_A g\,(t, x(t))\rho(t)dt$$

for every $x \in L^p(\mathcal{R}_+, \mu; \mathcal{R}^n)$ and $A \in \mathcal{L}$. Thus, g is another integrand that satisfies (b). The equality $f(t,\cdot) = g(t,\cdot)$ μ-a.e. $t \in \mathcal{R}_+$ readily follows from the theorem of Buttazzo and Dal Maso (1983).

To demonstrate that f is a Carathédory integrand, it suffices to show that f is a real-valued normal integrand. To this end, define

$$\widetilde{\Phi}(x, A) = \begin{cases} -I(x\chi_A) & \text{if } x\chi_A \in \mathcal{X}, \\ -\infty & \text{otherwise.} \end{cases}$$

Since $\widetilde{\Phi}$ satisfies the same conditions as Φ, from the preceding argument it follows that there exists a normal integrand $(-\tilde{f})\colon \mathcal{R}_+ \times \mathcal{R}^n \to \mathcal{R} \cup \{+\infty\}$ with $\tilde{f}(t, 0) = 0$ μ-a.e. $t \in \mathcal{R}_+$ such that (a) and (b) are true for $(\widetilde{\Phi}, \tilde{f})$. By the uniqueness of the integrand of I in the above argument, we have $f(t,\cdot) = -\tilde{f}(t,\cdot)$ μ-a.e. $t \in \mathcal{R}_+$, and hence, f is a normal integrand that does not take $\{\pm\infty\}$. Thus, the condition (i) is established.

To prove (2.2), suppose to the contrary that $\liminf_{t\to\infty} \rho\,(t) = \varepsilon > 0$. Then, there exists some $t_0 \in \mathcal{R}_+$ such that $\inf_{t \geq T} \rho\,(t) \geq \varepsilon$ for every $T \geq t_0$, and hence, $\rho(t) \geq \varepsilon$ for every $t \geq t_0$. Thus, we have $\mu([t_0, \infty)) = \int_{t_0}^\infty \rho\,(t)dt = \infty$, contradicting the fact that μ is a finite measure. □

Proof of Corollary 2.1

Suppose that $\Psi(z) = az + b$ for $z \in \mathcal{R}$ with $a > 0$ and $b \in \mathcal{R}$. Since $J = aI + b$ and $I(0) = J(0) = 0$, we have $b = 0$. By virtue of $v\chi_A \in \mathcal{X}$ and $J(v\chi_A) = aI(v\chi_A)$ for every $v \in X$ and $A \in \mathcal{L}$, we have $\int_A g\,(t, v)\rho(t)dt = \int_A a\,f(t, v)\rho(t)dt$ for every $A \in \mathcal{L}$, from which it follows that $g(t, v) = af(t, v)$ a.e. $t \in \mathcal{R}_+$ for every $v \in X$.

Conversely, let $g = af + b$ with $a > 0$ and $b \in \mathcal{R}$. Then $\Psi \circ I = aI + b\int_0^\infty \rho\,(t)dt$ and $\Psi(0) = b\int_0^\infty \rho\,(t)dt = b\mu(\mathcal{R}_+)$, from which it follows that $\Psi(z) = az + b\mu(\mathcal{R}_+)$ for every $z \in I(\mathcal{X})$. Define $\widetilde{\Psi}(z) = az + b\mu(\mathcal{R}_+)$ for $z \in \mathcal{R}$. We then obtain $J = \widetilde{\Psi} \circ I$. □

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[7] An $\mathcal{L} \times \mathcal{B}^n$-measurable function $(-f)\colon \mathcal{R}_+ \times \mathcal{R}^n \to \mathcal{R} \cup \{+\infty\}$ is a *normal integrand* if $f(t,\cdot)$ is upper semicontinuous on $\mathcal{R}^n$ μ-a.e. $t \in \mathcal{R}_+$ and $f(\cdot, v)$ is measurable on $\mathcal{R}_+$ for every $v \in \mathcal{R}^n$. Therefore, f is a Carathédory function if and only if both f and $-f$ are normal integrands (see Fonesca and Leoni 2007, Theorem 6.34).

2.4. Locally Constant Indifference of Preference Orderings

Axiom 2.8 (**Locally constant indifference**). For every $v \in X$ and $A, B \in \mathcal{L}$ with $0 < \mu(A) = \mu(B) < \infty$, $v\chi_A \sim v\chi_B$ whenever $v\chi_A, v\chi_B \in \mathcal{X}$.

Locally constant indifference (Axiom 2.8) is naturally satisfied for preference orderings defined by integral functionals with time-independent integrands. Consider, for simplicity, the preference ordering $\succsim$ on $L^p(\mathcal{R}_+, \mu; \mathcal{R}^n)$ defined by

$$x \succsim y \overset{\text{def}}{\Leftrightarrow} \int_0^\infty f(x(t))d\mu(t) \geq \int_0^\infty f(y(t))d\mu(t),$$

where $f: \mathcal{R}^n \to \mathcal{R}$ is a Borel measurable function satisfying $f(0) = 0$. Since $\int_0^\infty f(v\chi_A(t))d\mu(t) = f(v)\mu(A)$ for every $v \in \mathcal{R}^n$ and $A \in \mathcal{L}$ with $0 < \mu(A) < \infty$, locally constant indifference is valid for $(L^p(\mathcal{R}_+, \mu; \mathcal{R}^n), \succsim)$.

While this axiom is not used by Koopmans, his axioms of K-monotonicity (Axiom 2.6) and stationarity (Axiom 2.7) play an alternative role in deriving the independence of time indices for instantaneous utility functions with constant discount rates.

Theorem 2.2. *Suppose that the admissible set $\mathcal{X}$ is closed and connected in the L^p-norm topology and contains every locally constant trajectory. If $(\mathcal{X}, \succsim)$ satisfies Axioms 2.1 to 2.5 and 2.8, then the integrand f in Theorem 2.1 is autonomous on X, that is, f is independent of $t \in \mathcal{R}_+$ on X.*

Proof. Let $s, t \in \mathcal{R}_+ \setminus \{0\}$ with $s \neq t$ be arbitrary. Without loss of generality, we may assume that $s < t$. Choose $\varepsilon > 0$ sufficiently small such that the open interval $U_\varepsilon(s) = (s - \varepsilon, s + \varepsilon)$ is contained in $\mathcal{R}_+$. Since μ is nonatomic, there exists some $\varphi(\varepsilon) > 0$ with $\lim_{\varepsilon \to 0} \varphi(\varepsilon) = 0$ such that $\mu(U_\varepsilon(s)) = \mu(U_{\varphi(\varepsilon)}(t))$ with $U_{\varphi(\varepsilon)}(t) = (t - \varphi(\varepsilon), t + \varphi(\varepsilon)) \subset \mathcal{R}_+$. By locally constant indifference (Axiom 2.8), we have $v\chi_{U_\varepsilon(s)} \sim v\chi_{U_{\varphi(\varepsilon)}(t)}$ for every $v \in X$ and, hence, $\int_{U_\varepsilon(s)} f(\tau, v)d\mu(\tau) = I(v\chi_{U_\varepsilon(s)}) = I(v\chi_{U_{\varphi_\varepsilon}(t)}) = \int_{U_{\varphi(\varepsilon)}(t)} f(\tau, v)d\mu(\tau)$ for every $v \in X$. Note also that μ is a regular measure because of its absolute continuity with respect to the Lebesgue measure (see Halmos 1950, Sec.52(9)). Thus, by the Lebesgue–Besicovitch differentiation theorem (see Fonesca and Leoni 2007, Theorem 1.158), we have:

$$\begin{aligned} f(s, v) &= \lim_{\varepsilon \to 0} \frac{1}{\mu(U_\varepsilon(s))} \int_{U_\varepsilon(s)} f(\tau, v)d\mu(\tau) \\ &= \lim_{\varepsilon \to 0} \frac{1}{\mu(U_{\varphi(\varepsilon)}(t))} \int_{U_{\varphi(\varepsilon)}(t)} f(\tau, v)d\mu(\tau) = f(t, v). \end{aligned}$$

Therefore, $f(t, v)$ is constant a.e. $t \in \mathcal{R}_+$ for an arbitrarily fixed $v \in X$. □

Relations to the Savage Theory

The integral representation in Theorems 2.1 and 2.2 bears similarities to the Savage formulation for additive separable representations. Let $\mathcal{A}$ be an algebra of subsets of $\mathcal{R}_+$. The set of measurable functions $x: \mathcal{R}_+ \to X \subset \mathcal{R}^n$ with respect to $\mathcal{A}$ is denoted by $\mathcal{X}$. In Savage's terminology, $\mathcal{R}_+$ is a set of *states of the world*, X is a set of *consequences* and $\mathcal{X}$ is a set of extitacts.

The Savage axioms for $(\mathcal{X}, \succsim)$ guarantee that there exists a unique nonatomic finitely additive probability measure μ on $\mathcal{A}$ and a bounded function $f: X \to \mathcal{R}$ such that

$$x \succsim y \Leftrightarrow \int_0^\infty f\,(x(t))d\mu(t) \geq \int_0^\infty f\,(y(t))d\mu(t).$$

Here, the integrand f is unique up to a positive affine transformation (see Fishburn 1970, Chap.14). Since $\mathcal{X}$ contains L^p-spaces, the Savage formulation fits for our framework by the restriction of the preference ordering to the relevant choice space.

Ignoring the restriction that μ is supposed to be finitely additive and X is limited to a finite set in the Savage formulation, the main differences in the representation between Savage and this paper are the following: (i) μ is endogenously determined by axiomatization and represents a *qualitative probability* on $\mathcal{A}$ in the Savage model, while it is exogenously given in our model; (ii) μ does not involve a discount factor in the Savage model because it lacks the absolute continuity with respect to the Lebesgue measure, while given μ, a discount factor is uniquely determined in our model; (iii) the continuity of the preference ordering is not guaranteed under the Savage axioms; (iv) the growth condition on the integrand is derived in our model, which plays an especially crucial role in proving the existence of optimal paths in growth models without convexity assumptions (see Chichilnisky 1977 and Sagara 2001, 2007); (v) the integrand f is unique in the Savage model.

It should be mentioned that integral representations of this type are investigated in more general settings by Vind (2003, Chap.11 and 12 by Grodal). Unlike Savage (1972), Grodal and Vind treat the case where $\mathcal{A}$ is a σ-algebra and X is a metric space, but the measure μ is given arbitrarily. Under continuity, independence and translation invariance, they appear to succeed in obtaining a TAS representation for $(\mathcal{X}, \succsim)$ with exponential discounting.

One of the significant differences with this paper is a topological setting for the choice space. They endow $\mathcal{X}$ with the order topology with respect to $\succsim$. When restricting $\mathcal{X}$ to a subset of $L^p(\Omega, \mathcal{F}, \mu)$, their continuity requirement for $(\mathcal{X}, \succsim)$ is more stringent than ours in that they assume that the order topology on $\mathcal{X}$ is coarser than the L^p-norm topology. The uniqueness of the integrand and constant discountrate also seem unclear in their framework.

2.5. Convexity of Preference Orderings

In this subsection, we assume that the admissible set $\mathcal{X}$ is endowed with the weak topology of $L^p(\mathcal{R}_+, \mu; \mathcal{R}^n)$.

Axiom 2.1* (Weak continuity). For every $x \in \mathcal{X}$, the upper contour set $\{y \in \mathcal{X} | y \succsim x\}$ and the lower contour set $\{y \in \mathcal{X} | x \succsim y\}$ are weakly closed in $\mathcal{X}$.

Weak continuity (Axiom 2.1*) is a stricter requirement on preference orderings than strong continuity (Axiom 2.1).

Axiom 2.9 (Convexity). For every $x \in \mathcal{X}$, the upper contour set $\{y \in \mathcal{X} | y \succsim x\}$ is convex.

Note that if $\mathcal{X}$ is convex, then it is connected both in the norm topology and in the weak topology of $L^p(\Omega, \mathcal{F}, \mu)$.

Theorem 2.3. *Suppose that the admissible set $\mathcal{X}$ is convex, and closed in the weak topology of L^p. If $(\mathcal{X}, \succsim)$ satisfies Axioms 2.1* and 2.2 to 2.5, then it satisfies Axiom 2.9 and the integrand $f(t, \cdot)$ in Theorem 2.1 is concave μ-a.e. $t \in \mathcal{R}_+$.*

Proof. Weak continuity (Axiom 2.1*) implies that the utility function I for $(\mathcal{X}, \succsim)$ constructed in Lemma 2.1 is weakly continuous. Thus, the functional Φ defined in the proof of Theorem 2.1 is weakly upper semicontinuous on $L^p(\mathcal{R}_+, \mu; \mathcal{R}^n)$. The representation theorem of Buttazzo and Dal Maso (1983) guarantees the concavity of the integrand $f(t, \cdot)$ on $\mathcal{R}^n$, from which convexity (Axiom 2.9) follows. □

Theorem 2.3 is curious in that even if convexity (Axiom 2.9) is not assumed explicitly, weak continuity (Axiom 2.1*) necessarily implies convexity (Axiom 2.9), which is a consequence of strengthening the continuity requirement for the preference ordering. Convexity (Axiom 2.9), a geometric property, is obtainable from a topological property, weak continuity (Axiom 2.1*).

Linear Preference

Suppose that the admissible set $\mathcal{X}$ is given by (2.1). Let x^* be a continuous linear functional on $L^p(\mathcal{R}_+, \mu; \mathcal{R}^n)$ with $\langle x, x^* \rangle \geq 0$ for every $x \in \mathcal{X}$ and $\langle x, x^* \rangle = 0$ if and only if $x = 0$, where $\langle \cdot, \cdot \rangle$ denotes the duality between $L^p(\mathcal{R}_+, \mu; \mathcal{R}^n)$ and its dual space $(L^p(\mathcal{R}_+, \mu; \mathcal{R}^n))^*$.

Suppose that $\succsim$ is expressed by the restriction of x^* to $\mathcal{X}$, that is,

$$x \succsim y \overset{\text{def}}{\Longleftrightarrow} \langle x, x^* \rangle \geq \langle y, x^* \rangle.$$

It is evident that $(\mathcal{X}, \succsim)$ satisfies weak continuity (Axiom 2.1*), disjoint independence (Axiom 2.2) and disjoint additivity (Axiom 2.5).

Since $x \neq 0$ implies $\langle x, x^* \rangle > 0$, for every $A \in \mathcal{L}$ with a positive measure, it follows that $\langle x\chi_A, x^* \rangle > 0$ by choosing $x \in \mathcal{X}$ satisfying $x(t) > 0$ on A with $\mu(A) > 0$. Thus, $(\mathcal{X}, m)$ satisfies local sensitivity (Axiom 2.3) because of $x\chi_A \succ 0\chi_A$.

To show local substitutability (Axiom 2.4), take any $x \in \mathcal{X}$ and $A \in \mathcal{L}$ with a positive measure. Let $y \in \mathcal{X}$ be such that $y(t) > 0$ on A. We then have $\langle y\chi_A, x^* \rangle > 0$. Consider the continuous increasing function on $[0, \infty)$ defined by $\lambda \mapsto \langle \lambda y\chi_A, x^* \rangle$. Then, there exists some $\lambda \geq 0$ such that $\langle \lambda y\chi_A, x^* \rangle = \langle x, x^* \rangle$. Since $\mathcal{X}$ is a positive cone and $y\chi_A \in \mathcal{X}$, we have $\lambda y\chi_A \in \mathcal{X}$. This demonstrates local substitutability (Axiom 2.4).[8]

Therefore, by Theorem 2.1 (more precisely, $I(x) = \langle x, x^* \rangle$ works for the construction in the proof of Theorem 2.1), there exists a Carathéodory integrand $f: \mathcal{R}_+ \times \mathcal{R}^n \to \mathcal{R}$ such that $\langle x, x^* \rangle = \int_0^\infty f(t, x(t)) d\mu(t)$ for every $x \in \mathcal{X}$.

On the other hand, the Riesz representation theorem implies that there exists a unique $\varphi \in L^q(\mathcal{R}_+, \mu; \mathcal{R}^n)$ with $\frac{1}{p} + \frac{1}{q} = 1$ such that $\langle x, x^* \rangle = \int_0^\infty \langle x(t), \varphi(t) \rangle d\mu(t)$ for every $x \in \mathcal{X}$, where $\langle x(t), \varphi(t) \rangle$ is the inner product of $\mathcal{R}^n$. Thus, we have $\int_A f(t, x(t)) \rho(t) dt = \int_A \langle x(t), \varphi(t) \rangle \rho(t) dt$ for every $A \in \mathcal{L}$, where ρ is the Radon–Nikodym derivative of μ with respect to the Lebesgue measure, and, hence, $f(t, x(t)) = \langle x(t), \varphi(t) \rangle$ a.e. $t \in \mathcal{R}_+$ for every $x \in \mathcal{X}$. By the uniqueness of φ, we obtain $f(t, v) = \langle v, \varphi(t) \rangle$ a.e. $t \in \mathcal{R}_+$ for every $v \in \mathcal{R}^n_+$, and, hence, $f(t, 0) = 0$ a.e. $t \in \mathcal{R}_+$, where $\mathcal{R}^n_+$ is the nonnegative orthant of $\mathcal{R}^n$.

2.6. Selection of a Relevant Function Space

Theorems 2.1 to 2.3 are true for any choice of a complete Borel measure of the Lebesgue measurable space, $(\mathcal{R}_+, \mathcal{L})$, that is absolutely continuous with respect to the Lebesgue measure. In particular, the measure, μ_ρ, defined by (2.3), has the stated properties. Trajectories in

[8] For the axiomatization of linear preferences on a convex cone in L^1, see Weibull (1985).

$L^p(\mathcal{R}_+, \mu_\rho; \mathcal{R}^n)$ admit unboundedness under the norm of $L^p(\mathcal{R}_+; \mathcal{R}^n)$, but the growth rate of the paths is bounded by a function ρ. This type of space is called a *weighted L^p-space* with a weight function, ρ. Because there is some degree of freedom for the choice of ρ, one can obtain a TAS utility function with an exponential discount rate in Theorems 2.1 to 2.3 by choosing $\rho(t) = \exp(-rt)$ with $r > 0$.

As shown in the example below, in practice the choice of weight functions depends on the particular applications under investigation. If one wishes to treat an L^p-space as broadly as possible, it is desirable to choose a weight function as small as possible because $0 \leq \rho_1 \leq \rho_2$ implies that $L^p(\mathcal{R}_+, \mu_{\rho_2}; \mathcal{R}^n) \subset L^p(\mathcal{R}_+, \mu_{\rho_1}; \mathcal{R}^n)$.

We denote by $L^\infty(\mathcal{R}_+; \mathcal{R}^n)$ the set of essentially bounded functions on $\mathcal{R}_+$ to $\mathcal{R}^n$ with respect to the Lebesgue measure. If μ is a finite measure that is absolutely continuous with respect to the Lebesgue measure, then the following inclusion

$$L^\infty(\mathcal{R}_+; \mathcal{R}^n) \subset L^\infty(\mathcal{R}_+, \mu; \mathcal{R}^n) \subset L^p(\mathcal{R}_+, \mu; \mathcal{R}^n) \subset L^q(\mathcal{R}_+, \mu; \mathcal{R}^n)$$

is true for every $1 \leq q \leq p < \infty$. Instead of the ess. sup norm topology of $L^\infty(\mathcal{R}_+; \mathcal{R}^n)$, it is, thus, legitimate to endow $L^\infty(\mathcal{R}_+; \mathcal{R}^n)$ with the relative L^p-norm topology from $L^p(\mathcal{R}_+, \mu; \mathcal{R}^n)$. It is thereby possible to deal with a subset of $L^\infty(\mathcal{R}_+; \mathcal{R}^n)$ as an admissible set in $L^p(\mathcal{R}_+, \mu; \mathcal{R}^n)$.

Although $L^\infty(\mathcal{R}_+, \mu; \mathcal{R}^n)$ with the ess. sup norm is nonseparable, by equipping $L^\infty(\mathcal{R}_+; \mathcal{R}^n)$ with the L^p-norm topology, it follows from the separability of $L^p(\mathcal{R}_+, \mu; \mathcal{R}^n)$ that $L^\infty(\mathcal{R}_+; \mathcal{R}^n)$ becomes a separable Banach space. This is the reason why Theorems 2.1 to 2.3 are true in $L^\infty(\mathcal{R}_+; \mathcal{R}^n)$. Note that the restriction $1 \leq p < \infty$ is required here for $L^p(\mathcal{R}_+, \mu; \mathcal{R}^n)$ to apply the Debreu– Gorman theorem and the representation theorem of Buttazzo and Dal Maso (1983).

One Sector Optimal Growth with Nonconvex Technologies

We specify a relevant function space for standard one-sector optimal growth models. Let $\psi: \mathcal{R}_+ \times \mathcal{R}_+ \to \mathcal{R}_+$ be a production function and $\delta(t) \in [0,1]$ be a rate of depreciation of capital stock at$t \in \mathcal{R}_+$. A path of consumption is a Lebesgue measurable function $x: \mathcal{R}_+ \to \mathcal{R}$ generated by the ODE given by

$$x(t) = \psi\big(t, k(t)\big) - \delta(t)k(t) - \dot{k}(t) \quad \text{a.e. } t \in \mathcal{R}_+ \text{ and } k(0) = y, \tag{2.4}$$

where $k: \mathcal{R}_+ \to \mathcal{R}_+$, a locally absolutely continuous function, is a path of the capital stock. Define the set-valued mapping $\Gamma: \mathcal{R}_+ \times \mathcal{R}_+ \to 2^{\mathcal{R}}$ by

$$\Gamma(t, w) = \{v \in \mathcal{R} | -\delta(t)w \leq v \leq \psi(t, w) - \delta(t)w\}.$$

Given capital stock $w \geq 0$ at time $t \in \mathcal{R}_+$, the set of feasible consumption is denoted by $\Gamma(t, w)$. Thus, every solution to (2.4) corresponds to a solution to the differential inclusion

$$\dot{k}(t) \in \Gamma(t, k(t)) \quad \text{a.e. } t \in \mathcal{R}_+ \text{ and } k(0) = y. \tag{2.5}$$

Suppose that the following conditions are satisfied.

(i) ψ is continuous on $\mathcal{R}_+ \times \mathcal{R}_+$, $\psi(t, \cdot)$ is increasing and $\psi(t, 0) = 0$ for every $t \in \mathcal{R}_+$.

(ii) There exists a continuous function $\gamma: \mathcal{R}_+ \to \mathcal{R}_+$ and $a > 0$ such that

$$\psi(t, v) \leq \gamma(t) + av^p \quad \text{for every } (t, v) \in \mathcal{R}_+ \times \mathcal{R}_+$$

and

$$\int_0^\infty \exp\left(-\int_0^t \gamma(s) ds\right) dt < \infty.$$

The Peano existence theorem (Hartman 1982, Theorem II.2.1) shows that any solution to $\dot{k}(t) = \psi(t, k(t))$ with $k(0) = y$ on $[0, T)$ can be extended to $[0, \infty)$, which we denote by $k(t|y)$. In the terminology of capital theory, $k(t|y)$is a "pure accumulation path." Since $\psi(t, \cdot)$ is increasing, this solution is unique (Hartman 1982, Corollary III.6.3). It follows that if $\dot{k}(t) \leq \psi(t, k(t))$ with $k(0) = y$, then $k(t) \leq k(t|y)$ (see Hartman 1982, Theorem III.4.1). Let $\alpha(t) = \max\{k(t|y), \gamma(t) + ak(t|y)^p\}$. It follows that $\dot{k}(t) \in \Gamma(t, k(t))$ a.e. $t \in \mathcal{R}_+$ implies $\max\{|k(t)|, |\dot{k}(t)|\} \leq \alpha(t)$ a.e. $t \in \mathcal{R}_+$.

Define the weight function, ρ, by

$$\rho(t) = \frac{exp\,(-\int_0^t \gamma\,(s)ds)}{1 + \alpha(t)^p}.$$

Note that ρ is determined exclusively by the production technology. Since ρ is positive, continuous and Lebesgue integrable on $\mathcal{R}_+$, the measure μ_ρ defined by (2.3) is a complete, nonatomic, finite, regular Borel measure. By condition (ii), we have

$$\int_0^\infty \alpha\,(t)^p \rho(t)dt \leq \int_0^\infty \exp\left(-\int_0^t \gamma\,(s)ds\right)dt < \infty$$

and, hence, $\alpha \in L^p(\mathcal{R}_+, \mu_\rho) \subset L^1(\mathcal{R}_+, \mu_\rho)$. Note also that γ belongs to $L^1(\mathcal{R}_+, \mu_\rho)$ in view of $\gamma \leq \alpha$. Therefore, every locally absolutely continuous function $k: \mathcal{R}_+ \to \mathcal{R}_+$ that is a solution to (2.5) is such that k and $\dot{k}$ are in $L^p(\mathcal{R}_+, \mu_\rho)$. For every solution (x, k) to (2.4), we have

$$|x(t)| \leq (\gamma(t) + ak(t)^p) + \delta(t)k(t) + |\dot{k}(t)| \leq 3\alpha(t) \quad \text{a.e. } t \in \mathcal{R}_+,$$

and, hence, $x \in L^1(\mathcal{R}_+, \mu_\rho)$. This suggests that the suitable selection for the admissible set is a subset of $L^1(\mathcal{R}_+, \mu_\rho)$.

3. TAS Representation of Recursive Utility

3.1. Recursive Preference Orderings

In this section we investigate the TAS representability of a general form of recursive utility functionals that permits nonexponential variable discount factors. An essential feature of recursive utility is that the rate of time preference is endogenized in its structure. Recursive utility was formulated by Koopmans (1960) in a discrete time framework. It was Uzawa (1968) who extended the Koopmans discrete time concept of recursive utility to continuous time. Epstein (1987) axiomatized a "generating function," by which recursive utility functionals are obtained as a solution of an ODE. The existence of optimal paths in convex growth models with recursive utility was investigated by Becker et al. (1989) and the existence of those in nonconvex growth modelswith recursive utility by Sagara (2001).

Let ρ be a Lebesgue integrable continuous function on $\mathcal{R}_+$ with positive values and define the measure μ_ρ by (2.3). Suppose that the choice space $\mathcal{X}$ is a subset of $L^p(\mathcal{R}_+, \mu_\rho; \mathcal{R}^n)$ with $1 \leq p < \infty$, but the admissibility of $\mathcal{X}$ is no longer required in the sequel. An integral functional I is given by a recursive utility functional of the form:

$$I(x) = \int_0^\infty \left[L(t, x(t))F\left(t, \int_0^t g\left(s, x(s)\right)ds\right)\right]dt \quad \text{for } x \in \mathcal{X}. \tag{3.1}$$

Here, L and g are $\mathcal{L} \times \mathcal{B}^n$-measurable functions on $\mathcal{R}_+ \times \mathcal{R}^n$ and F is a Lebesgue measurable function on $\mathcal{R}_+ \times \mathcal{R}$. A recursive preference ordering $\succsim$ on $\mathcal{X}$ is defined by:

$$x \succsim y \overset{\text{def}}{\Longleftrightarrow} I(x) \geq I(y). \tag{3.2}$$

The integral functional form of recursive utility (3.1) is very general. A typical case in economic applications appears when L and g are autonomous and $F(t,z) = \exp(-z)$, which is the case studied by Epstein (1987a, 1987b). If, moreover, $L \equiv -1$, the case is reduced to Epstein and Hynes (1983). Uzawa (1968) investigated the case for $F(t,z) = \exp(-z)$ and $g(t,v) = \theta(L(v))$ with an increasing function θ. Because dynamic inconsistency stems from nonexponential discounting, as emphasized by Strotz (1955), the recursive utility (3.1) can easily incorporate the dynamically inconsistent behaviors of decision makers.

3.2. TAS Representation II

Assumption 3.1.

(i) $L(t,\cdot)$ is continuous on $\mathcal{R}^n$ and $L(\cdot,v)$ is Lebesgue measurable on $\mathcal{R}_+$ for every $v \in \mathcal{R}^n$.

(ii) There exist some $\alpha \in L^1(\mathcal{R}_+,\mu_\rho)$ and $a > 0$ such that

$$|L(t,v)| \leq \alpha(t) + a|v|^p \quad \text{for every } (t,v) \in \mathcal{R}_+ \times \mathcal{R}^n.$$

(iii) $F(t,\cdot)$ is continuous on $\mathcal{R}$ a.e. $t \in \mathcal{R}_+$ and $F(\cdot,z)$ is Lebesgue measurable on $\mathcal{R}_+$ for every $z \in \mathcal{R}$.

(iv) $g(t,\cdot)$ is continuous on $\mathcal{R}^n$ a.e. $t \in \mathcal{R}_+$ and $g(\cdot,v)$ is Lebesgue measurable on $\mathcal{R}_+$ for every $v \in \mathcal{R}^n$.

(v) There exists some $\theta \in L^1_{\text{loc}}(\mathcal{R}_+,\mu_\rho)$ such that

$$|g(t,v)| \leq \theta(t) \quad \text{a.e. } t \in \mathcal{R}_+ \text{ for every } v \in \mathcal{R}^n$$

and

$$\left| F\left(t, \int_0^t \theta(s)ds\right) \right| \leq \rho(t) \quad \text{a.e. } t \in \mathcal{R}_+.$$

(vi) $L(t,v)F(t,z) \leq 0$ a.e. $t \in \mathcal{R}_+$ for every $(v,z) \in \mathcal{R}^n \times \mathcal{R}$.

As proved in Lemma 3.1, the recursive utility functional I is upper semicontinuous on $\mathcal{X}$ in the L^p-norm topology under Assumption 3.1. Moreover, $(\mathcal{X},\succsim)$ defined by (3.2) satisfies disjoint independence (Axiom 2.2) and disjoint additivity (Axiom 2.5) whenever $\mathcal{X}$ is admissible.

Assumption 3.2.

(i) $L(t,v) \leq 0$ a.e. $t \in \mathcal{R}_+$ for every $v \in \mathcal{R}^n$.

(ii) $F(t,z) \geq 0$ a.e. $t \in \mathcal{R}_+$ for every $z \in \mathcal{R}$ and $F(t,\cdot)$ is decreasing on $\mathcal{R}$ a.e. $t \in \mathcal{R}_+$.

(iii) $L(t,\cdot)F(t,\cdot)$ is concave on $\mathcal{R}^n \times \mathcal{R}$ a.e. $t \in \mathcal{R}_+$.

(iv) $g(t,\cdot)$ is concave on $\mathcal{R}^n$ a.e. $t \in \mathcal{R}_+$.

As demonstrated in Theorem 3.2, I is concave on $\mathcal{X}$ under Assumption 3.2 whenever $\mathcal{X}$ is convex. Thus, $(\mathcal{X},\succsim)$ satisfies convexity (Axiom 2.8).

Theorem 3.1. *Let $I\colon \mathcal{X} \to \mathcal{R}$ be a recursive utility functional defined by (3.1). If $\mathcal{X}$ is closed in the L^p-norm topology and Assumption 3.1 is satisfied, then there exists an $\mathcal{L} \times \mathcal{B}^n$-measurable* *function* $f: \mathcal{R}_+ \times \mathcal{R}^n \to \mathcal{R} \cup \{-\infty\}$ *satisfying* $I(x) = \int_0^\infty f(t, x(t))\rho(t)dt$ *for* $x \in \mathcal{X}$ *with the following properties:*

(i) $f(t,\cdot)$ *is upper semicontinuous on* $\mathcal{R}^n$ *a.e.* $t \in \mathcal{R}_+$ *and* $f(\cdot, v)$ *is Lebesgue measurable on* $\mathcal{R}_+$ *for every* $v \in \mathcal{R}^n$.

(ii) *There exist some* $\alpha \in L^1(\mathcal{R}_+, \mu_\rho)$ *and* $\beta \geq 0$ *such that*

$$f(t, v) \leq \alpha(t) + \beta|v|^p \quad a.e.\ t \in \mathcal{R}_+ \ for\ every\ v \in \mathcal{R}^n.$$

Moreover, if g *is another integrand with* $I = \int_0^\infty g\,\rho dt$ *that satisfies condition (ii) with* $g(t, 0) = 0$ *a.e.* $t \in \mathcal{R}_+$, *then* $f(t,\cdot) = g(t,\cdot)$ *a.e.* $t \in \mathcal{R}_+$. *Furthermore, if* $\mathcal{X}$ *is closed in the weak topology of* L^p *and convex, and Assumption 3.2 is satisfied, then* $f(t,\cdot)$ *is a concave integrand a.e.* $t \in \mathcal{R}_+$.

Contrary to Theorem 2.1, local sensitivity (Axiom 2.3) and local substitutability (Axiom 2.4) need not be fulfilled for the recursive preference ordering (3.2) to obtain a TAS representation by means of Theorem 3.1. This observation suggests that only strong continuity (Axiom 2.1), disjoint independence (Axiom 2.2) and disjoint additivity (Axiom 2.5) are possibly sufficient for obtaining a TAS representation in Theorem 2.1.

Theorem 3.1 demonstrates that recursive utility representations can be reduced to TAS utility representations and leads to the significant implication from the point of view of applications that in dynamic optimization problems, simple econometric tests cannot distinguish the shape of the discount function $F(t, \int_0^t g(s, x(s))ds)$ of a decision maker, that is, whether the decision maker's discount function depends on the whole history of trajectories or not. It is well known from Strotz (1955) that time-varying discount factors may lead to time-inconsistent optimal decisions. Thus, it is impossible to determine if the decision maker is a dynamically consistent or a dynamically inconsistent optimizer.

Continuity of the Recursive Utility

Lemma 3.1. *If Assumption 3.1 is satisfied, then* I *is upper semicontinuous on* $\mathcal{X}$ *in the* L^p*-norm topology.*

Proof. The argument is based on Sagara (2007). It is easy to verify that by the growth conditions (ii) and (v) of Assumption 3.1, we have:

$$\left| L(t, x(t)) F\left(t, \int_0^t g(s, x(s))ds\right) \right| \leq (\alpha(t) + a|x(t)|^p)\rho(t) \tag{3.3}$$

for every $x \in \mathcal{X}$ a.e. $t \in \mathcal{R}_+$ and the right-hand side of the inequality (3.3) is Lebesgue integrable over $\mathcal{R}_+$ for every $x \in \mathcal{X}$. Thus, I is well defined. Define the Nemytskii operator $T: L^p(\mathcal{R}_+, \mu_\rho; \mathcal{R}^n) \to L^1(\mathcal{R}_+, \mu_\rho)$ by $(Tx)(t) = L(t, x(t))$. By condition (ii) of Assumption 3.1, there exist some $\alpha \in L^1(\mathcal{R}_+, \mu_\rho)$ and $a > 0$ such that $|(Tx)(t)| \leq \alpha(t) + a|x(t)|^p$ for every $x \in L^p(\mathcal{R}_+, \mu_\rho; \mathcal{R}^n)$ and $t \in \mathcal{R}_+$. Thus, $Tx \in L^1(\mathcal{R}_+, \mu_\rho)$ is well defined for every $x \in L^p(\mathcal{R}_+, \mu_\rho; \mathcal{R}^n)$. Since μ_ρ is nonatomic because of the nonatomicity of the Lebesgue measure and L is a Carathéodory function, it follows that T is bounded and continuous (see Krasnosel'skii 1964, Theorem 2.1).

Define the function R on $\mathcal{R}_+ \times L^p(\mathcal{R}_+, \mu_\rho; \mathcal{R}^n)$ by

$$R(t,x) = F\left(t, \int_0^t g\,(s, x(s))ds\right).$$

We show that R is a Carathéodory function, that is, $R(t,\cdot)$ is continuous on $L^p(\mathcal{R}_+, \mu_\rho; \mathcal{R}^n)$ a.e. $t \in \mathcal{R}_+$ and $R(\cdot, x)$ is Lebesgue measurable on $\mathcal{R}_+$ for every $x \in L^p(alR_+, \mu_\rho; \mathcal{R}^n)$. Note first that F is a Carathéodory function by condition (iii) of Assumption 3.1 and, hence, it is jointly measurable on $\mathcal{R}_+ \times \mathcal{R}$ (see Fonesca and Leoni 2007, Theorem 6.34). Thus, the measurability of $R(\cdot, x)$ for every $x \in L^p(\mathcal{R}_+, \mu_\rho; \mathcal{R}^n)$ is immediate. Let $\{x^\nu\}$ be a convergent sequence in $L^p(\mathcal{R}_+, \mu_\rho; \mathcal{R}^n)$ to some x. Then, $\{x^\nu\}$ has a subsequence (which we do not relabel) such that $x^\nu(t) \to x(t)$ μ_ρ-a.e. $t \in \mathcal{R}_+$. Since the Lebesgue measure is absolutely continuous with respect to μ_ρ by the positivity of ρ, we have $x^\nu(t) \to x(t)$ a.e. $t \in \mathcal{R}_+$. Since $|g(t, x^\nu(t))| \le \theta(t)$ a.e. $t \in \mathcal{R}_+$ for each ν and $g(t, x^\nu(t)) \to g(t, x(t))$ a.e. $t \in \mathcal{R}_+$ by conditions (iv) and (v) of Assumption 3.1, we have:

$$\lim_{\nu\to\infty} \int_0^t g\,(s, x^\nu(s))ds = \int_0^t g\,(s, x(s))ds \quad \text{for every } t \in \mathcal{R}_+$$

by the Lebesgue dominated convergence theorem. Therefore, we have:

$$\lim_{\nu\to\infty} F\left(t, \int_0^t g\,(s, x^\nu(s))ds\right) = F\left(t, \int_0^t g\,(s, x(s))ds\right) \quad \text{a.e. } t \in \mathcal{R}_+.$$

Therefore, $R(t,\cdot)$ is continuous a.e. $t \in \mathcal{R}_+$.

Define the operator, $\Psi: \mathcal{X} \to L^1(\mathcal{R}_+)$, by

$$(\Psi x)(t) = L(t, x(t))F\left(t, \int_0^t g\,(s, x(s))ds\right).$$

By (3.3), Ψx is integrable over $\mathcal{R}_+$ for every $x \in \mathcal{X}$ and by condition (vi) of Assumption 3.1, $\Psi x \le 0$ for every $x \in \mathcal{X}$. Let $\{x^\nu\}$ be a convergent sequence in $\mathcal{X}$to some $x \in \mathcal{X}$. Since $Tx^\nu \to Tx$ in $L^1(\mathcal{R}_+, \mu_\rho)$, the sequence $\{Tx^\nu\}$ has a subsequence (which we do not relabel) such that $(Tx^\nu)(t) \to (Tx)(t)$ μ_ρ-a.e. $t \in \mathcal{R}_+$, and hence, $(Tx^\nu)(t) \to (Tx)(t)$ a.e. $t \in \mathcal{R}_+$. Since $(\Psi x^\nu)(t) = (Tx^\nu)(t)R(t, x^\nu)$ for each ν and $(\Psi x^\nu)(t) \to (\Psi x)(t)$ a.e. $t \in \mathcal{R}_+$, Fatou's lemma implies:

$$\limsup_{\nu\to\infty} I\,(x^\nu) \le \int_0^\infty \limsup_{\nu\to\infty} (\,\Psi x^\nu)(t)dt = \int_0^\infty (\,\Psi x)(t)dt = I(x).$$

Therefore, I is upper semicontinuous on $\mathcal{X}$. □

Concavity of the Recursive Utility

Lemma 3.2. *Suppose that $\mathcal{X}$ is convex. If Assumption 3.2 is satisfied, then I is concave on $\mathcal{X}$.*

Proof. Let $x_0, x_1 \in \mathcal{X}$ and $\lambda \in [0,1]$ be arbitrary. Define the functions on $\mathcal{R}_+$ by $z_0(t) = \int_0^t g\,(s, x_0(s))ds$ and $z_1(t) = \int_0^t g\,(s, x_1(s))ds$. It follows from Assumption 3.2 that:

$$L(t, \lambda x_0(t) + (1-\lambda)x_1(t))F\left(t, \int_0^t g\,(s, \lambda x_0(s) + (1-\lambda)x_1(s))ds\right)$$
$$\geq L(t, \lambda x_0(t) + (1-\lambda)x_1(t))F(t, \lambda z_0(t) + (1-\lambda)z_1(t))$$
$$\geq \lambda L(t, x_0(t))F(t, z_0(t)) + (1-\lambda)L(t, x_1(t))F(t, z_1(t)),$$

a.e. $t \in \mathcal{R}_+$, where the second line uses conditions (i), (ii) and (iv) of Assumption 3.2 and the third line employs condition (iii) of Assumption 3.2. Therefore, integrating this inequality yields $I(\lambda x_0 + (1-\lambda)x_1) \geq \lambda I(x_0) + (1-\lambda)I(x_1)$. □

Proof of Theorem 3.1

Define $\Phi: L^p(\mathcal{R}_+, \mu_\rho; \mathcal{R}^n) \times \mathcal{L} \to \mathcal{R} \cup \{-\infty\}$ by:

$$\Phi(x, A) = \begin{cases} \int_A [L(t, x(t))F(t, \int_0^t g\,(s, x(s))ds)]dt & \text{if } x \in \mathcal{X}_A, \\ -\infty & \text{otherwise.} \end{cases}$$

Since $\Phi(\cdot, \mathcal{R}_+)$ is upper semicontinuous on the closed set $\mathcal{X}$ in the L^p-norm topology by Lemma 3.1, it follows that $\Phi(\cdot, \mathcal{R}_+)$ is upper semicontinuous on $L^p(\mathcal{R}_+, \mu_\rho; \mathcal{R}^n)$ by construction. Note that Φ is countably additive and local, and satisfies $\Phi(0, A) = 0$ for every $A \in \mathcal{L}$. Thus, Φ satisfies the conditions of the representation theorem by Buttazzo and Dal Maso (1983) (see also the proof of Theorem 2.1). Therefore, there exists a normal integrand $(-f): \mathcal{R}_+ \times \mathcal{R}^n \to \mathcal{R} \cup \{+\infty\}$ with the following properties:

(a) There exist some $\alpha \in L^1(\mathcal{R}_+, \mu_\rho)$ and $\beta \geq 0$ such that

$$f(t, v) \leq \alpha(t) + \beta|v|^p \quad \mu_\rho\text{-a.e. } t \in \mathcal{R}_+ \text{ for every } v \in \mathcal{R}^n.$$

(b) $\Phi(x, A) = \int_A f\,(t, x(t))d\mu_\rho(t)$ for every $x \in L^p(\mathcal{R}_+, \mu_\rho; \mathcal{R}^n)$ and $A \in \mathcal{L}$.

Thus, conditions (i) and (ii) of the theorem are established. The condition (b) implies that $I(x) = \int_0^\infty f\,(t, x(t))\rho(t)dt$ for every $x \in \mathcal{X}$. The argument for the uniqueness of the integrand is same as the proof of Theorem 2.1.

Since $\Phi(\cdot, \mathcal{R}_+)$ is concave and upper semicontinuous on the Banach space $L^p(\mathcal{R}_+, \mu_\rho; \mathcal{R}^n)$ by Lemmas 3.1 and 3.2, it is also weakly upper semicontinuous on $L^p(\mathcal{R}_+, \mu_\rho; \mathcal{R}^n)$ (see Fonesca and Leoni 2007, Proposition 4.26). From the representation theorem by Buttazzo and Dal Maso (1983), it follows that $f(t, \cdot)$ is concave on $\mathcal{R}^n$ a.e. $t \in \mathcal{R}_+$. □

4. Concluding Remarks

In this paper, a TAS representation of preference orderings with an infinite horizon in continuous time has been formulated under a different system of axioms to that of Koopmans (1972b). In particular, this paper has not imposed the continuous time analogue of his axioms, K-monotonicity (Axiom 2.6) and stationarity (Axiom 2.7), which jointly imply the time-invariance of preference orderings and play an important role in determining discount rates endogenously in the Koopmans argument.

The unique discount function derived in this paper is not determined axiomatically, but just as a consequence of the absolute continuity of the underlying measure. Thus, it is desirable to obtain a corresponding TAS representation in continuous time, hopefully under the same topological setting on a choice space (L^∞ with the ess. sup norm), which might make a direct comparison possible between the axioms in continuous time and those in discrete time.

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