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A Direct Proof of Aumann and Maschlers' Theorem on The Nucleolus of A Bankruptcy Game

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Abstract

An alternative proof of Aumann and Maschlers' theorem on the nucleolus of a Talmudic bankruptcy game is given directly from the definition of the nucleolus.

1. Introduction

The purpose of this note is to give a direct proof of Aumann and Maschlers' interesting theorem on a bankruptcy problem based on the Talmud [1]. This theorem states that the CG-consistent solution to a bankruptcy problem, which is defined after a Talmudic principle called by them the *contested garment principle,* is precisely the nucleolus of a game associated with the bankruptcy problem.

Their short and elegant proof makes use of theorems of cooperative game theory, e. g., [2], [3], [4], some of them being not so familiar to non-specialists. The proof is completed by showing that the kernel of the associated game consists of a single point, thereby establishing the identity of it and the nucleolus via the theorem of Schmeidler [4].

In contrast, in the proof to be given below, we use only the definition of the nucleolus [4], which makes the proof rather lenghy, yet direct, quite elementary and more easily accessible for non-specialists.

We give only the definitions and results that are necessary for our proof. For motivations and discussions on them, refer to Aumann and Maschler [1]. The proof is performed in a straightforward manner by first representing formally the CG-consistent solution which Aumann and Maschler have described in their theorem A, and then showing directly that no other solution can satisfy the requirement of the nucleolus.

2. Definitions and the Theorem

A bankruptcy problem is a pair $(E; d)$ where E is the estate of a bankrupt, and $d=(d_1, \ldots, d_n)$, $0 \leqq d_1 \leqq \ldots \leqq d_n$, is the debts to n creditors $1, \ldots, n$, satisfying $0 \leqq E \leqq d_1+\ldots+d_n \equiv D$. A solution to $(E; d)$ is an n-tuple $x=(x_1, \ldots, x_n)$ of real numbers with $x_1+\ldots+x_n=E$.

A solution x is called *CG-consistent* if for all $i \neq j$, (x_i, x_j) satisfies

$$x_i=(X_{ij}-(X_{ij}-d_i)_+-(X_{ij}-d_j)_+)/2+(X_{ij}-d_j)_+$$

and

$$x_j=(X_{ij}-(X_{ij}-d_i)_+-(X_{ij}-d_j)_+/2+(X_{ij}-d_i)_+,$$

where

$$X_{ij} \equiv x_i+x_j,$$

$$t_+ \equiv \max(t, 0).$$

Aumann and Maschler [1] have shown that every bankruptcy problem has a unique CG-consistent solution.

In this note, a *game* is a function v that associates a nonnegative real number $v(S)$ with each subset S of $N=\{1, \ldots, n\}$. N is the set of *players*, and S is called a *coalition*. It is assumed that $v(\phi)=0$. A *payoff vector* is a vector $x=(x_1, \ldots, x_n)$ with $x_1+\ldots+x_n=v(N)$, where x_i represents a payoff to player i. An imputation is a payoff vector x satisfying $x_i \geqq v(\{i\})$ for all $i \in N$.

A *bankruptcy game* associated with a bankruptcy problem $(E; d)$ is a game $v_{E;d}$ defined by

$$v_{E;d}(S)=(E-d(N-S))_+ \text{ for each } S \subset N,$$

where

$$z(R) \equiv \sum_{i \in R} z_i$$

for any $R \subset N$ and any vector $z=(z_1, \ldots, z_n)$.

The nucleolus of a game v is an imputation x obtained as follows [4]. For a given imputation y, let $\theta(y)$ be a vector in R^{2^n}, the 2^n-dimensional Euclidean space, the components of which are the numbers $v(S)-y(S)$ for all subsets S arranged in the non-increasing order, i. e., $\theta_1(y) \geqq \theta_2(y) \geqq \ldots \geqq \theta_{2^n}(y)$. Then, an imputation x is called the nucleolus of v if for any imputation $y \neq x$,

$$\theta_{i_0}(x) < \theta_{i_0}(y)$$

where

$$i_0 \equiv \min\{h | \theta_h(x) \neq \theta_h(y)\}.$$

It is well known that every game v has a unique nucleolus [4].

The number $v(S)-y(S)$ is called the excess of coalition S with respect to y. Thus, the nucleolus has the meaning that it minimizes the maximal excess among the coalitions.

A striking result about the nucleolus of the bankruptcy game is that it is precisely the CG-consistent solution of the bankruptcy problem. Namely,

Theorem (Aumann and Maschler [1]). The CG-consistent solution of a bankruptcy problem (E, d) is the nucleolus of the game $v_{E;d}$.

To prove this theorem from the definition of the nucleolus, we need an explicit representation of the CG-consistent solution. Let x be the CG-consistent solution. Then, following the construction in Theorem A of Aumann and Maschler [1], x can be given as follows:

Case (i) If $0 \leqq E \leqq nd_1/2$, then

$$x_i = E/n \text{ for all } i=1, \ldots, n.$$

Case (*ii*) For $k=0, 1, 2, \ldots, n-2$, if

$$(D-\sum_{j=k+1}^{n}(d_j-d_{k+1}))/2\leqq E\leqq(D-\sum_{j=k+2}^{n}(d_j-d_{k+2}))/2,$$

then

$$x_i=d_i/2 \text{ for } i=1, \ldots, k+1$$
$$x_i=c_{k+1} \text{ for } i=k+2, \ldots, n,$$

where

$$c_{k+1}=d_{k+1}/2+\{E-(D-\sum_{j=k+1}^{n}(d_j-d_{k+1}))/2\}/(n-k-1).$$

In this case we have

$$x_i\leqq d_i/2, \text{ for } i=k+2, \ldots, n.$$

To see this, put $E=(D-\sum_{j=k+2}^{n}(d_j-d_{k+2}))/2$ for each k to obtain $c_{k+1}\leqq d_{k+2}/2$.

Case (*iii*) For $k=n-2, n-3, \ldots, 1, 0$, if

$$(D+\sum_{j=k+2}^{n}(d_j-d_{k+2}))/2\leqq E\leqq(D+\sum_{j=k+1}^{n}(d_j-d_{k+1}))/2,$$

then

$$x_i=d_i/2 \text{ for } i=1, \ldots, k+1$$
$$x_i=d_i-b_{k+1} \text{ for } i=k+2, \ldots, n,$$

where

$$b_{k+1}=d_{k+1}/2+\{(D+\sum_{j=k+1}^{n}(d_j-d_{k+1}))/2-E\}/(n-k-1).$$

In this case we have

$$x_i\geqq d_i/2, \text{ for } i=k+2, \ldots, n.$$

To see this, put $E=(D+\sum_{j=k+2}^{n}(d_j-d_{k+2}))/2$ for each k to obtain $b_{k+1}\leqq d_{k+2}/2$.

Case (*iv*) If $D-nd_1/2\leqq E\leqq D$, then

$$x_i=d_i-(D-E)/n \text{ for all } i=1, \ldots, n.$$

It will be convenient to note that the four cases are arranged in the increasing order of E from 0 to D.

3. Proofs

Initially, we state four easy lemmas. The bankruptcy game $v_{E;d}$ will be denoted simply by v.

Lemma 1. If $E \leqq (D-(d_n - d_{n-1}))/2$, then

$$v(\{i\})=0 \text{ for all } i=1, \ldots, n.$$

Proof. Note that $D-d_n \geqq d_{n-1}$. Then,

$$E \leqq (D-d_n)/2+d_{n-1}/2 \leqq D-d_n$$

Hence, for all i,

$$0 \leqq v(\{i\})=\max\{0, E-D+d_i\}$$
$$\leqq \max\{0, E-D+d_n\}=v(\{n\})=0.$$

Lemma 2. If $(D+(d_n-d_{n-1}))/2 \leqq E$, then

$$v(N-\{i\})=E-d_i \text{ for all } i=1, \ldots, n.$$

Proof.

$$E \geqq d_n/2+(D-d_{n-1})/2 \geqq d_n.$$

Hence, $E \geqq d_i$ for all $i=1, \ldots, n$, which implies

$$v(N-\{i\})=\max\{0, E-d_i\}=E-d_i, \text{ for all } i=1, \ldots, n.$$

Lemma 3. If

$$(D-\sum_{j=k+1}^{n}(d_j-d_{k+1}))2 \leqq E,$$

then

$$v(N-\{i\})=E-d_i \text{ for all } i=1, \ldots, k+1.$$

Proof.

$$E \geqq D/2-\sum_{j=k+1}^{n}(d_j-d_{k+1})/2$$
$$=(\sum_{j=1}^{n} d_j+\sum_{j=k+1}^{n} d_{k+1})/2 \geqq 2d_{k+1}/2,$$

because $k \leqq n-2$. Hence, for all $i=1, \ldots, k+1$,

$$E-d_i \geqq E-d_{k+1} \geqq 0,$$

which implies

$$v(N-\{i\})=\max\{0, E-d_i\}=E-d_i \text{ for all } i=1, \ldots, k+1.$$

Lemma 4. If

$$E \leqq (D+\sum_{j=k+1}^{n}(d_j-d_{k+1})),$$

then

$$v(\{i\})=0 \text{ for all } i=1, \ldots, k+1.$$

Proof.

$$E \leqq D/2+(D-\sum_{j=1}^{n} d_j-\sum_{j=k+1}^{n} d_{k+1})/2$$

$$=D-(\sum_{j=1}^{n} d_j+\sum_{j=k+1}^{n} d_{k+1})/2$$

$$\leqq D-2d_{k+1}/2=D-d_{k+1},$$

because $k \leqq n-2$. Hence, for all $i=1, \ldots, k+1$,

$$E-D+d_i \leqq E-D+d_{k+1}$$

which implies

$$v(\{i\})=0 \text{ for all } i=1, \ldots, k+1.$$

We now prove the theorem. Cases (i) and (iv) are proved before cases (ii) and (iii). In all proofs, S will stand for a nonempty, proper subset of N. The values $v(S)-x(S)$ for $S=\phi$ or N are always 0, so that they can be ignored.

Case (i) If $0 \leqq E \leqq nd_1/2$, then

$$x_i=E/n \text{ for all } i=1, \ldots, n.$$

Proof. We show that if $S \subset N$, $S \neq N$ then

$$v(S)-x(S) \leqq v(\{i\})-x_i=-E/n \text{ for all } i=1, \ldots, n.$$

Note that

$$E \leqq nd_1/2=(D-\sum_{j=1}^{n}(d_j-d_1))/2 \leqq (D-(d_n-d_{n-1}))/2.$$

It then follows from Lemma 1 that

$$v(\{i\})=0 \text{ for all } i=1,\dots,n.$$

Then, if $v(S)=0$, there is a j such that

$$v(S)-x(S)\leqq v(\{j\})-x_j=v(\{i\})-x_i=-E/n$$

for all $i=1,\dots,n$. If $v(S)>0$, then noting that $x_i\leqq d_i$ for all $i=1,\dots,n$, we have for some j,

$$\begin{aligned} v(S)-x(S)&\leqq v(N-\{j\})-x(N-\{j\})=-d_j+x_j \\ &=-d_j+E/n\leqq -E/n \\ &=-x_i \text{ for all } i=1,\dots,n. \end{aligned}$$

Thus, the assertion is true. This also implies that x is an imputation.

Now, let y be any payoff vector with $y\neq x$. Then, for some i, we must have $y_i<x_i$. Hence,

$$v(\{i\})-y_i>v(\{i\})-x_i=-E/n \text{ for this } i,$$

which implies that y is not the nucleolus.

Case (*iv*) If $D-nd_1/2\leqq E\leqq D$, then

$$x_i=d_i-(D-E)/n \text{ for all } i=1,\dots,n.$$

Proof. We show that if $S\subset N$, $S\neq N$ then

$$v(S)-x(S)\leqq v(N-\{i\})-x(N-\{i\})=-(D-E)/n$$

for all $i=1,\dots,n$. Note that

$$E\geqq D-nd_1/2=(D+\sum_{j=1}^{n}(d_j-d_1))/2\geqq(D+(d_n-d_{n-1}))/2.$$

It then follows from Lemma 2 that

$$v(N-\{i\})=E-d_i \text{ for all } i=1,\dots,n.$$

If $v(S)>0$, then noting that $x_i\leqq d_i$ for all $i=1,\dots,n$, we have for some j,

$$\begin{aligned} v(S)-x(S)&\leqq v(N-\{j\})-x(N-\{j\})=-d_j+x_j \\ &=-(D-E)/n=-d_i+x_i \\ &=v(N-\{i\})-x(N-\{i\}) \text{ for all } i=1,\dots,n. \end{aligned}$$

If $v(S)=0$, then for some j we have

$$v(S)-x(S)\leqq v(\{j\})-x_j$$

$$=-d_j+(D-E)/n \leqq -(D-E)/n$$
$$=v(N-\{i\})-x(N-\{i\}) \text{ for all } i=1, \ldots, n.$$

Thus, the assertion is true and x is an imputation.

Now, let y be any payoff vector with $y \neq x$. Then, for some i, we must have $y_i > x_i$. Hence,

$$v(N-\{i\})-y(N-\{i\})>v(N-\{i\})-x(N-\{i\}) \text{ for this } i,$$

which implies that y is not the nucleolus.

Case (*ii*) For $k=0, 1, 2, \ldots, n-2$, if

$$(D-\sum_{j=k+1}^{n}(d_j-d_{k+1}))/2 \leqq E \leqq (D-\sum_{j=k+2}^{n}(d_j-d_{k+2}))/2,$$

then

$$x_i=d_i/2 \text{ for } i=1, \ldots, k+1$$
$$x_i=c_{k+1} \text{ for } i=k+2, \ldots, n,$$

where

$$c_{k+1}=d_{k+1}/2+\{E-(D-\sum_{j=k+1}^{n}(d_j-d_{k+1}))/2\}/(n-k-1)$$

Proof. Assume first that $k \leqq n-3$. Then, $E \leqq (D-(d_n-d_{n-1}))/2$. Hence, by Lemma 1,

$$v(\{i\})=0 \text{ for all } i=1, \ldots, n.$$

Also, by Lemma 3,

$$v(N-\{i\})=E-d_i \text{ for all } i=1, \ldots, k+1.$$

We show that for each $i=1, \ldots, k$, if S satisfies

$S \neq \{1\}$, $S \neq N-\{1\}$, $S \neq \{2\}$, $S \neq N-\{2\}, \ldots, S \neq \{i\}$, and $S \neq N-\{i\}$,

then

$$v(S)-x(S) \leqq v(N-\{i+1\})-x(N-\{i+1\})$$
$$=v(\{i+1\})-x_{i+1}$$
$$=-d_{i+1}/2 \qquad (1)$$

Recall that $x_i \leqq d_i/2 < d_i$ for all $i=1, 2, \ldots, n$. Then, for any such S, we have:

$v(S)>0$ implies $\exists h_{i+1} \neq 1, 2, \ldots, i$ such that

$$v(S)-x(S) \leqq v(N-\{h_{i+1}\})-x(N-\{h_{i+1}\})$$
$$=-d_{h_{i+1}}+x_{h_{i+1}} \leqq -x_{h_{i+1}},$$

and

$$v(S)=0 \text{ implies } \exists j_{i+1}\neq 1,2,\dots,i \text{ such that}$$
$$v(S)-x(S)\leqq v(\{j_{i+1}\})-x_{j_{i+1}}$$
$$=-x_{j_{i+1}}.$$

But, by Lemmas 3 and 1, we have

$$v(N-\{i+1\})-x(N-\{i+1\})=-d_{i+1}+x_{i+1}$$
$$=-d_{i+1}/2=-x_{i+1}\geqq -x_{h_{i+1}}$$

and

$$v(\{i+1\})-x_{i+1}=0-x_{i+1}=-d_{i+1}/2\geqq -x_{j_{i+1}}.$$

Hence, (1) holds.

We next show that if S satisfies

$$S\neq\{1\},\ S\neq N-\{1\},\dots,S\neq\{k+1\},\ S\neq N-\{k+1\},$$

then

$$v(S)-x(S)\leqq v(\{j\})-x(\{j\})$$
$$=-c_{k+1}\leqq -d_{k+1}/2 \text{ for all } j=k+2,\dots,n-1,n. \quad (2)$$

This is because we have:

$$v(S)=0 \text{ implies } v(S)-x(S)\leqq v(\{j\})-x_j$$
$$=-x_j=-c_{k+1}\leqq -d_{k+1}/2$$

and

$$v(S)>0 \text{ implies } v(S)-x(S)\leqq v(N-\{j\})-x(N-\{j\})$$
$$=-d_j+x_j\leqq -x_j\leqq -d_{k+1}/2.$$

Combining (1) and (2), and noting that

$$v(\{1\})-x_1=-d_1/2=v(N-\{1\})-x(N-\{1\}),$$

we conclude that the first n greatest values of $v(S)-x(S)$ can be arranged in the non-increasing order as

$$-d_1/2\geqq -d_2/2\geqq\dots\geqq -d_{k+1}/2\geqq -c_{k+1}=\dots=-c_{k+1} \quad (3)$$

which also implies that x is an imputation.

Now, let y be any payoff vector with $y\neq x$, and let

$$i_0=\min\{i|y_i\neq x_i\}.$$

Then, if $i_0\leqq k+1$, it follows from (1) that

$$v(N-\{i_0\})-y(N-\{i_0\})>-d_{i_0}/2$$

or

$$v(\{i_0\})-y_{i_0}>-d_{i_0}/2.$$

Hence, y cannot be the nucleolus. If $i_0 \geqq k+2$, then due to the assumption that $k \leqq n-3$, there is another $j_0 \geqq k+2$ such that

$$y_{i_0} < x_{i_0} \text{ implies } y_{j_0} > x_{i_0}$$

and

$$y_{i_0} > x_{i_0} \text{ implies } y_{j_0} < x_{i_0}$$

Hence, we must have either

$$v(\{i_0\}) - y_{i_0} > -c_{k+1},$$

or

$$v(\{j_0\}) - y_{j_0} > -c_{k+1},$$

which implies that y is not the nucleolus.

When $k=n-2$, we have

$$x_i = d_i/2 \qquad i=1, 2, \dots, n-1,$$
$$x_n = c_n \geqq d_{n-1}/2$$

and (3) now becomes

$$-d_1/2 \geqq -d_2/2 \geqq \dots -d_{n-1}/2 \geqq -c_n.$$

Note that $i_0 < n$ by definition. Hence, it follows from (1) that either

$$v(\{i_0\}) - y_{i_0} > -d_{i_0}/2$$

or

$$v(N-\{i_0\}) - y(N-\{i_0\}) > -d_{i_0}/2,$$

which implies that y is not the nucleolus. This completes the proof.

Case (iii) For $k=n-2, n-3, \dots, 1, 0,$ if

$$(D+\sum_{j=k+2}^{n}(d_j-d_{k+2}))/2 \leqq E \leqq (D+\sum_{j=k+1}^{n}(d_j-d_{k+1}))/2,$$

then

$$x_i = d_i/2 \text{ for } i=1, \dots, k+1$$
$$x_i = d_i - b_{k+1} \text{ for } i=k+2, \dots, n,$$
$$b_{k+1} = d_{k+1}/2 + \{D+\sum_{j=k+1}^{n}(d_j-d_{k+1}))/2 - E\}/(n-k-1).$$

Proof. The proof is similar to case (ii). Assume first that $k \leqq n-3$. Then, $E \geqq (D+(d_n-d_{n-1}))/2$. Hence, by Lemma 2,

$$v(N-\{i\}) = E - d_i \text{ for all } i=1, \dots, n.$$

Also, by Lemma 4,

$$v(\{i\})=0 \text{ for all } i=1, \ldots, k+1.$$

We show that for each $i=1, \ldots, k$, if S satisfies

$$S \neq \{1\},\ S \neq N-\{1\},\ S \neq \{2\},\ S \neq N-\{2\}, \ldots, S \neq \{i\}, \text{ and } S \neq N-\{i\},$$

then

$$\begin{aligned} v(S)-x(S) &\leqq v(N-\{i+1\})-x(N-\{i+1\}) \\ &= v(\{i+1\})-x_{i+1} \\ &= -d_{i+1}/2 \qquad (1') \end{aligned}$$

Recall that $d_i/2 \leqq x_i \leqq d_i$ for all $i=1, 2, \ldots, n$. Then, for any such S, we have:

$$v(S)>0 \text{ implies } \exists h_{i+1} \neq 1, 2, \ldots, i \text{ such that}$$

$$\begin{aligned} v(S)-x(S) &\leqq v(N-\{h_{i+1}\})-x(N-\{h_{i+1}\}) \\ &= -d_{h_{i+1}}+x_{h_{i+1}} \\ &= -d_{h_{i+1}}/2 \quad \text{if } h_{i+1} \leqq k+1 \\ &= -b_{k+1} \quad \text{if } h_{i+1} \geqq k+2 \end{aligned}$$

and

$$v(S)=0 \text{ implies } \exists j_{i+1} \neq 1, 2, \ldots, i \text{ such that}$$

$$\begin{aligned} v(S)-x(S) &\leqq v(\{j_{i+1}\})-x_{j_{i+1}} \\ &= -x_{j_{i+1}} \end{aligned}$$

But, by lemmas 2 and 4, we have

$$\begin{aligned} v(N-\{i+1\})-x(N-\{i+1\}) &= -d_{i+1}+x_{i+1} \\ &= -d_{i+1}/2 \geqq -d_{i+2}/2 \geqq \ldots \\ &\geqq -d_{k+1}/2 \geqq -b_{k+1}, \end{aligned}$$

and

$$v(\{i+1\})-x_{i+1}=0-x_{i+1}=-d_{i+1}/2 \geqq -x_{j_{i+1}}.$$

Hence, (1′) holds.

We next show that if S satisfies

$$S \neq \{1\},\ S \neq N-\{1\}, \ldots, S \neq \{k+1\},\ S \neq N-\{k+1\},$$

then

$$\begin{aligned} v(S)-x(S) &\leqq v(N-\{j\})-x(N-\{j\}) \\ &= -b_{k+1} \leqq -d_{k+1}/2 \text{ for all } j=k+2, \ldots, n-1, n. \qquad (2') \end{aligned}$$

This is because we have:

$$v(S)>0 \text{ implies } v(S)-x(S) \leqq v(N-\{j\})-x(N-\{j\})$$

$$= -d_j + x_j = -b_{k+1} \leqq -d_{k+1}/2.$$

and

$$v(S)=0 \text{ implies } v(S)-x(S) \leqq v(\{j\})-x_j$$
$$= -x_j \leqq -d_j + x_j$$
$$= -b_{k+1} \leqq -d_{k+1}/2$$

Combining (1′) and (2′), and noting that

$$v(\{1\})-x_1 = -d_1/2 = v(N-\{1\})-x(N-\{1\}),$$

we conclude that the first n greatest values of $v(S)-x(S)$ can be arranged in the non-increasing order as

$$-d_1/2 \geqq -d_2/2 \geqq \ldots \geqq -d_{k+1}/2 \geqq -b_{k+1} = \ldots = -b_{k+1} \qquad (3')$$

which also implies that x is an imputation. The rest of the proof is almost the same to that of case (ii), so is omitted.

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