A comment on Ries [Ries, R., 2007, The analytics of monetary non-neutrality in the Sidrauski model, Economics Letters 94 (1), 129-135]

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A comment on Ries [Ries, R., 2007, The analytics of monetary non-neutrality in the Sidrauski model, Economics Letters 94 (1), 129-135]

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December 8, 2010

Abstract

This note gives a counterexample on Ries [Ries, R., 2007, The analytics of monetary non-neutrality in the Sidrauski model, Economics Letters 94 (1), 129–135]. Using a certain family of utility functions, this note not only gives a sharper representation than that of Ries but also demonstrates that interest rate inelastic money demand does not lead to superneutrality. This implies that superneutrality does not exist when uncertainty is introduced.

Keywords: monetary policy, superneutrality, nominal interest rate policy, perfect complementary between consumption and money
JEL classifications: E5, O42

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1 Introduction

Ries (2007) characterized the dynamics of the money-in-the-utility model (Sidrauski, 1967) by using the money demand function to explain the mechanism in a very intuitive manner. One of his main conclusions is that when assuming that the government can control nominal interest rates by setting any growth rate of money supply, monetary policy does not affect any level of consumption and capital stock as long as either money demand is inelastic with respect to nominal interest or money and consumption are separable in the utility function. Subsequently, Lioui and Poncet (2008) attached uncertainty with Ries' framework to demonstrate that superneutrality is valid only in the case of an interest rate inelastic money demand. However, both studies do not pursue a sufficient investigation on the relationship between the money demand function and the utility function.

This note gives a counterexample for their propositions. That is, we show that within a certain family of utility functions, interest rate inelastic money demand does not lead to superneutrality. An intuitive explanation is as follows. A nominal interest monetary policy affects real variables through the product of the interest rate elasticity of money demand and the elasticity of the marginal utility of consumption with respect to money. When consumption and money are perfectly complementary, the former elasticity is zero but the latter elasticity takes infinity. When the product of both elasticities converges to a finite value, such a policy is still effective.

2 Ries–Sidrauski Economy

In order to prepare a counterexample, this section briefly reviews a Ries–Sidrauski economy and reconsiders the assumptions on the utility function of Ries (2007).

In the economy, $c_t > 0$, $k_t > 0$, and $m_t > 0$, respectively, denote consumption, capital stock, and real balances or just money. Technology is characterized by a constant parameter $\delta > 0$ of depreciation rate and a production function $f(k_t) < 0$ with $f_k > 0$, $f_{kk} < 0$, $f(0) = 0$, $\lim_{k \to 0} f_k = \infty$, and $\lim_{k \to \infty} f_k = 0$. Representative agents are infinity lived with perfect foresight, and their preferences are characterized by a constant parameter $\rho > 0$ of the rate of time preference and a utility function $u(c_t, m_t)$. A set of assumptions imposed on u is discussed later.

In equilibrium, the representative agent maximizes their lifetime utility to choose c_t and m_t, the markets are clear, and the government chooses nominal interest rates $R_t = f_k(k_t) + \pi_t$, where π_t denotes inflation rates, by controlling an appropriate rate of money growth.

The equilibrium dynamics system is characterized by the money demand function $\varphi(c, R)$, defined by $R = u_m(c, \varphi)/u_c(c, \varphi)$, which results from the necessary condition for the maximization problem of the representative agents. Using φ, we describe the dynamics system[1] as

$$\theta\frac{\dot{c}}{c} = f_k - \delta - \rho - \xi\eta\frac{\dot{R}}{R}$$
$$\dot{k} = f - \delta k - c.$$

where $\theta = -cu_{cc}(c,\varphi(c,R))/u_c(c,\varphi(c,R)) - \xi\zeta$, $\xi = mu_{cm}(c,\varphi(c,R))/u_c(c,\varphi(c,R))$, $\eta = -R\varphi_R(c,R)/\varphi(c,R)$, and $\zeta = c\varphi_c(c,R)/\varphi(c,R)$, respectively, represent the inverse of the intertemporal elasticity of substitution, the elasticity of the marginal utility of consumption with respect to real money balances, the interest rate elasticity of money demand, and the consumption elasticity of money demand.

Ries (2007), in his proposition 2, stated that money is superneutral when $\xi\eta$ is equal to zero. Following the proposition, Ries stated that such superneutrality attains either if money and consumption are separable in the utility function ($\xi = 0$) or if money demand is inelastic with respect to nominal interest ($\eta = 0$). In this note, we give a counterexample satisfying $\eta = 0$ but $\xi\eta \neq 0$.

Before providing the example, we discuss a set of assumptions regarding the utility function. Ries (2007) assumed $u_c > 0$, $u_{cc} < 0$, $u_m \geq 0$, $u_{mm} \leq 0$, $u_{cc}u_{nn} \geq u_{cm}$, and $u_{cm} \geq 0$. When we assume $u_m = 0$, then $u_m/u_c = R = 0$, implying that the government should set zero nominal interest rates. In addition, when we assume $u_{cc}u_{mm} = u_{cm}^2$, then, as shown later, we cannot exclude the possibility of $\theta = 0$.

[1] In the conventional monetary policy with a constant rate of money growth μ, we should add

$$\eta\frac{\dot{R}}{R} + \zeta\frac{\dot{c}}{c} = f_k + \mu - R$$

to the two equations in order to describe the system.

The assumption $u_{cm} \geq 0$ is a little bit restrictive because this assumption excludes the case of $\theta > 1$ in the famous CRRA form of $u(c,m) = (c^{1-\alpha}m^{\alpha})^{1-\theta}/(1-\theta)$, where $0 < \alpha < 1$ is a constant parameter.

Instead of the above assumptions on the utility function, we propose the following assumption: $u_c > 0$, $u_{cc} < 0$, $u_m > 0$, $u_{mm} < 0$, $u_{cc}u_{nn} - u_{cm}^2 > 0$, $u_{cm}u_m - u_{mm}u_c > 0$, and $u_c u_{cm} - u_{cc}u_m > 0$ for all $c > 0$ and $m > 0$. The first four assumptions indicate that u is strictly increasing and strictly concave with respect to c and m. The last two assumptions arise from $\partial(u_m/u_c)/\partial c > 0$ and $\partial(u_m/u_c)/\partial m > 0$. These assumptions are the same as those of Fischer (1979). Using the total differential form $dR = \{\partial(u_m/u_c)/\partial c\}dc + \{\partial(u_m/u_c)/\partial m\}dm$, we obtain

$$\begin{aligned} \varphi_R &= \left. \frac{u_c^2}{u_{mm}u_c - u_{cm}u_m} \right|_{m=\varphi(c,R)} \\ \varphi_c &= \left. \frac{u_{cc}u_m - u_c u_{cm}}{u_{mm}u_c - u_{cm}u_m} \right|_{m=\varphi(c,R)} \end{aligned} \tag{1}$$

and

$$\theta = -c \left. \frac{u_{cc}u_{mm} - u_{cm}^2}{u_{mm}u_c - u_{cm}u_m} \right|_{m=\varphi(c,R)}.$$

Therefore, if the above assumptions are satisfied, then $-\varphi_R$, φ_c, and θ are all nonnegative. When $u_{cm}u_m - u_{mm}u_c$ and $u_c u_{cm} - u_{cc}u_m$ are finite, then $-\varphi_R$, φ_c, and θ are all positive.

From equation (1), the interest rate elasticity of money demand $\eta = -R\varphi_R/\varphi$ might takes zero only when $u_{cm}u_m - u_{mm}u_c$ takes infinity, This would happen when u_{cm} or $\zeta = mu_{cm}/u_c$ takes infinity. This makes us conjecture that, even when $\eta = 0$, the product of η and ζ is not necessarily zero.

3 Counterexample

Because we cannot prove the conjecture in the above general class of utility functions, we set a somewhat restrictive class to give a counterexample. Let $u(c,m) = w(c\psi(z))$, where $z = m/c > 0$. When $-c\psi w''/w'$ is constant, this is exactly the class of utility functions Lucas (2000) proposed. In order for u to be strictly increasing and strictly concave with respect to c and m, respectively, we assume that w and ψ are strictly increasing and strictly concave, respectively, and $0 < z\psi'/\psi < 1$ for all $z > 0$.

Under this class, the money demand function is determined by

$$R = \frac{\psi(z)}{\psi(z) - z\psi'(z)}. \tag{2}$$

The right-hand side of the above equation is positive and strictly decreasing for all $z > 0$,[2] and, accordingly, there exists an inverse function $z = \phi(R)$. Thus, the money demand function $m = c\phi(R)$ is well-defined. The elasticities of the money demand function with respect to consumption and interest rates are, respectively, unity and

$$\eta = -\frac{R\phi'(R)}{\phi(R)} = -\left.\frac{\psi'(z)\{\psi(z) - z\psi'(z)\}}{z\psi(z)\psi''(z)}\right|_{z=\phi(R)}.$$

[2] In fact

$$\frac{d}{dz}\frac{\psi(z)}{\psi(z) - z\psi'(z)} = \frac{\psi''(z)\psi(z)}{\{\psi(z) - z\psi'(z)\}^2} < 0.$$

The last equality is established by using equation (1) and $u(c,m) = w(c\psi(z))$.

The dynamic is described as the same in the previous section and the coefficients are expressed in a simpler way. With some algebraic operations,[3] we can get $\theta = c\psi w''/w'|_{z=\phi(R)}$ and

$$\xi = (1/\eta - \theta)\frac{z\psi'(z)}{\psi(z)}\bigg|_{z=\phi(R)}. \tag{3}$$

Equation (3) indicates that the elasticity of the shadow price u_c with respect to money is represented much more clearly than that of Ries (2007). That is, the elasticity ξ is determined by η, θ, and the relative slope of ψ. When $1/\eta > \theta$ or $\eta < 1/\theta$, then the interest elasticity of money demand is smaller than the elasticity of the intertemporal substitution. In this case, the shadow price of capital is increasing in money. When $\eta = 1/\theta$, then $u_{cm} = 0$ or the utility function is separable.

Because $\xi\eta = (1-\eta\theta)z\psi'/\psi$ and $0 < z\psi'/\psi < 1$, we can show $\eta = 0$ but $\xi\eta \neq 0$ within our family of utility function. Even if $\eta \to 0$, ξ is growing much faster, and, accordingly, $\xi\eta$ converges to $z\psi'/\psi$. Only when the utility function is separable does $\xi\eta$ take the value of zero.

Finally, we present a parametric example. The utility function is described as

$$u(c,m) = \frac{[(1-\alpha)c^{\frac{\eta-1}{\eta}} + \alpha m^{\frac{\eta-1}{\eta}}]^{\frac{\eta(1-\theta)}{\eta-1}}}{1-\theta} = \frac{\{c\psi(z)\}^{1-\theta}}{1-\theta}$$

where $0 < \alpha < 1$, $\theta > 0$, and $\eta \geq 0$ are constant parameters and $\psi(z) = [1 -$

[3]See Appendix.

$\alpha + \alpha z^{\frac{\eta-1}{\eta}}]^{\frac{\eta}{\eta-1}}$ for $z = m/c > 0$. Notice that $z = \min[c, m]$ when $\eta = 0$ and that $z = c^{1-\alpha} m^{\alpha}$ when $\eta = 1$. Consumption and real balances are perfect complements when $\eta = 0$. The case of $\eta = 0$ corresponds the case of a cash-in-advance economy, in which money is needed for purchasing consumption goods and the cash-in-advance constraint is always binding.[4]

In this case, the elasticity of intertemporal substitution and the interest rate elasticity are respectively determined by the constant parameters $1/\theta$ and η, and $\xi\eta$ is represented as a function only of R, or

$$\xi\eta = \left(\frac{\alpha^{\eta} R^{1-\eta}}{(1-\alpha)^{\eta} + \alpha^{\eta} R^{1-\eta}} \right) (1 - \theta\eta).$$

Clearly, $\xi\eta = R/(1+R)$ when $\eta = 0$ and $\xi\eta = \alpha(1-\theta)$ when $\eta = 1$. When $\theta\eta = 1$, $\xi\eta$ takes zero.

4 Concluding Remarks

In summary, using a larger set of utility functions than that of Lucas (2000), we not only give a sharper representation than that of Ries (2007) but also give a counterexample. When consumption and real balances are perfectly complement, then the interest rate elasticity of money demand is zero but a nominal interest policy is not superneutral. Only in the case of a separable utility function does superneutrality survive. This discussion assumes that consumers have perfect foresight and

[4]The constraint $m \leq c$ is binding when the government sets the nominal interest rate to be positive.

no uncertainty exists. When uncertainty is introduced, following Lioui and Poncet (2008), separability does not assure superneutrality. Therefore, no superneutrality exists with our family of utility functions.

Appendix

Consider $u(c,m) = w(y)$, where $y = \psi(m/c)c$. The derivatives of u are described as follows:

$$\begin{aligned}
u_c &= \{\psi(m/c) - \psi'(m/c)m/c\}w'(y) \\
u_m &= \psi'(m/c)w'(y) \\
u_{cc} &= \{\psi(m/c) - \psi'(m/c)m/c\}^2 w''(y) + (m^2/c^3)\psi''(m/c)w'(y) \\
u_{mm} &= \{\psi'(m/c)\}^2 w''(y) + \psi''(m/c)(1/c)w'(y) \\
u_{cm} &= \psi'(m/c)\{\psi(m/c) - \psi'(m/c)m/c\}w''(y) - (m/c^2)\psi''(m/c)w'(y).
\end{aligned}$$

The money demand function is derived from equation (2). The total differential form is described as $dR = \{\partial(u_m/u_c)/\partial c\}dc + \{\partial(u_m/u_c)/\partial m\}dm$, where

$$\begin{aligned}
\frac{\partial(u_m/u_c)}{\partial c} &= \frac{u_c u_{cm} - u_m u_{cc}}{u_c^2} = -\frac{(m/c^2)\psi(m/c)\psi''(m/c)}{\{\psi(m/c) - (m/c)\psi'(m/c)\}^2} \\
\frac{\partial(u_m/u_c)}{\partial m} &= \frac{u_c u_{mm} - u_m u_{cm}}{u_c^2} = \frac{(1/c)\psi(m/c)\psi''(m/c)}{\{\psi(m/c) - (m/c)\psi'(m/c)\}^2}.
\end{aligned}$$

Using $\varphi_R = 1/\{\partial(u_m/u_c)/\partial m\}$ and $\varphi_c = -\{\partial(u_m/u_c)/\partial c\}/\{\partial(u_m/u_c)/\partial m\} =$

m/c, we obtain:

$$\begin{aligned}
\eta &= = -R/\{m\partial(u_m/u_c)/\partial m\} \\
&= -\frac{\psi'(m/c)\{\psi(m/c) - (m/c)\psi'(m/c)\}}{(m/c)\psi(m/c)\psi''(m/c)} \\
\zeta &= -\{c\partial(u_m/u_c)/\partial c\}/\{m\partial(u_m/u_c)/\partial m\} = 1.
\end{aligned}$$

Because of $\zeta = 1$,

$$\theta = -\frac{cu_{cc}}{u_c} - \frac{mu_{cm}}{u_c} = -\frac{c\psi(m/c)w''(y)}{w'(y)}.$$

Because of

$$\frac{mu_{cm}}{u_c} = -\theta\frac{m/c\psi'(m/c)}{\psi(m/c)} - \frac{(m^2/c^2)\psi''(m/c)}{\psi(m/c) - \psi'(m/c)m/c},$$

we obtain equation (3).

References

[1] Fischer, S., 1979, Capital accumulation on the transition path in a monetary optimizing model, Econometrica 47, 6, 1433–1439.

[2] Lioui, A., and P. Poncet, 2008, Monetary non-neutrality in the Sidrauski model under uncertainty, Economics Letters 100 (1), 22–26.

[3] Lucas Jr., R. E., 2000, Inflation and welfare, Econometrica 68, 247–274.

[4] Ries, R., 2007, The analytics of monetary non-neutrality in the Sidrauski model, Economics Letters 94 (1), 129–135.

[5] Sidrauski, M., 1967, The rational choice and patterns of growth in a monetary economy, American Economic Review 57, 534–544.

Technical appendix to A remark on Ries (2007)

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December 8, 2010

When considering a general utility function $u(c,m)$, the money demand function $\varphi(c,R)$, defined by $R = u_m(c,\varphi)/u_c(c,\varphi)$. The dynamics of the Ries economy are described as

$$\begin{aligned} \xi\eta\frac{\dot{R}}{R} + \theta\frac{\dot{c}}{c} &= f_k - \delta - \rho \\ \dot{k} &= f - \delta k - c, \end{aligned}$$

where $\theta = -cu_{cc}/u_c - \xi\zeta$, $\xi = mu_{cm}/u_c$, $\eta = -R\varphi_R/\varphi$, and $\zeta = c\varphi_c/\varphi$.

As for u, we assume: $u_c > 0$, $u_{cc} < 0$, $u_m > 0$, $u_{mm} < 0$, $u_{cc}u_{nn} - u_{cm}^2 > 0$, $u_{cm}u_m - u_{mm}u_c > 0$, and $u_c u_{cm} - u_{cc}u_m > 0$ for all $c > 0$ and $m > 0$. The last two assumptions arise from $\partial(u_m/u_c)/\partial c > 0$ and $\partial(u_m/u_c)/\partial m > 0$. Using the total differential form, $dR = \{\partial(u_m/u_c)/\partial c\}dc + \{\partial(u_m/u_c)/\partial m\}dm$, we obtain

$$\varphi_R = \frac{u_c^2}{u_{mm}u_c - u_{cm}u_m} \tag{1}$$

$$\varphi_c = \frac{u_{cc}u_m - u_c u_{cm}}{u_{mm}u_c - u_{cm}u_m} \tag{2}$$

and

$$\begin{aligned} \theta &= -cu_{cc}/u_c - u_{cm}c\varphi_c/u_c \\ &= -cu_{cc}/u_c - \frac{cu_{cm}(u_{cc}u_m - u_c u_{cm})}{u_c(u_{mm}u_c - u_{cm}u_m)} \end{aligned}$$

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$$
\begin{aligned}
&= -\frac{c}{u_c}\frac{u_{cc}(u_{mm}u_c - u_{cm}u_m) + u_{cm}(u_{cc}u_m - u_c u_{cm})}{u_{mm}u_c - u_{cm}u_m} \\
&= -c\frac{u_{cc}u_{mm} - u_{cm}^2}{u_{mm}u_c - u_{cm}u_m}.
\end{aligned}
$$

Consider a more restrictive family of utility functions such that $u(c,m) = w(y)$, where $y = \psi(m/c)c$. We assume: $w' > 0$, $w'' < 0$, $\psi > 0$, $\psi' > 0$, $\psi'' < 0$, and $\psi(m/c) - \psi'(m/c)m/c > 0$. The derivatives of u are described as follows:

$$
\begin{aligned}
u_c &= \{\psi(m/c) - \psi'(m/c)m/c\}w'(y) \\
u_m &= \psi'(m/c)w'(y) \\
u_{cc} &= \{\psi(m/c) - \psi'(m/c)m/c\}^2 w''(y) \\
&\quad +(m^2/c^3)\psi''(m/c)w'(y) \\
u_{mm} &= \{\psi'(m/c)\}^2 w''(y) \\
&\quad +\psi''(m/c)(1/c)w'(y) \\
u_{cm} &= \psi'(m/c)\{\psi(m/c) - \psi'(m/c)m/c\}w''(y) \\
&\quad -(m/c^2)\psi''(m/c)w'(y).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\frac{cu_{cc}}{u_c} &= \frac{\{c\psi(m/c) - m\psi'(m/c)\}w''(y)}{w'(y)} \\
&\quad +\frac{(m^2/c^2)\psi''(m/c)}{\psi(m/c) - \psi'(m/c)m/c} \\
\frac{mu_{cm}}{u_c} &= \frac{m\psi'(m/c)w''(y)}{w'(y)} \\
&\quad -\frac{(m^2/c^2)\psi''(y)}{\psi(m/c) - \psi'(m/c)m/c} \\
\frac{cu_{cc}}{u_c} + \frac{mu_{cm}}{u_c} &= \frac{c\psi(m/c)w''(y)}{w'(y)} \\
u_{cc}u_{mm} - u_{cm}^2 &= [\{\psi - \psi'm/c\}^2 w'' + (m^2/c^3)\psi''w'][\{\psi'\}^2 w'' + \psi''(1/c)w'] \\
&\quad -[\psi'\{\psi - \psi'm/c\}w'' - (m/c^2)\psi''w']^2 \\
&= \{\psi - \psi'm/c\}^2\{\psi'\}^2\{w''\}^2 + (m^2/c^3)(1/c)\{\psi''\}^2\{w'\}^2 \\
&\quad +\{\psi^2 - 2\psi\psi'm/c + (\psi')^2(m/c)^2\}(1/c)\psi''w''w' + (m^2/c^3)\psi''\{\psi'\}^2 w''w' \\
&\quad -\{\psi - \psi'm/c\}^2\{\psi'\}^2\{w''\}^2 - (m^2/c^4)\{\psi''\}^2\{w'\}^2
\end{aligned}
$$

$$
\begin{aligned}
& +2\psi'\{\psi - \psi' m/c\}(m/c^2)\psi'' w'' w' \\
= \quad & \{\psi^2 - 2\psi\psi' m/c + 2(\psi')^2 (m/c)^2\}(1/c)\psi'' w'' w' \\
& +2\psi'\{\psi - \psi' m/c\}(m/c^2)\psi'' w'' w' \\
= \quad & (\psi^2/c)\psi'' w'' w'.
\end{aligned}
$$

The money demand function is derived from

$$R = \frac{\psi(z)}{\psi(z) - z\psi'(z)},$$

where $z = m/c$. The right-hand side of the above equation is strictly decreasing because of

$$\frac{d}{dz}\frac{\psi(z)}{\psi(z) - z\psi'(z)} = \frac{\psi''(z)\psi(z)}{\{\psi(z) - z\psi'(z)\}^2} < 0.$$

Accordingly, there exists an inverse function $z = \phi(R)$. Thus, the money demand function $m = c\phi(R) = \varphi$ is well-defined.

The total differential form is described as $dR = \{\partial(u_m/u_c)/\partial c\}dc + \{\partial(u_m/u_c)/\partial m\}dm$, where

$$
\begin{aligned}
\frac{\partial(u_m/u_c)}{\partial c} &= \frac{u_c u_{cm} - u_m u_{cc}}{u_c^2} = -\frac{m\psi(m/c)\psi''(m/c)}{\{c\psi(m/c) - m\psi'(m/c)\}^2} \\
&= -\frac{(m/c^2)\psi(m/c)\psi''(m/c)}{\{\psi(m/c) - (m/c)\psi'(m/c)\}^2} \\
\frac{\partial(u_m/u_c)}{\partial m} &= \frac{u_c u_{mm} - u_m u_{cm}}{u_c^2} = \frac{c\psi(m/c)\psi''(m/c)}{\{c\psi(m/c) - m\psi'(m/c)\}^2} \\
&= \frac{(1/c)\psi(m/c)\psi''(m/c)}{\{\psi(m/c) - (m/c)\psi'(m/c)\}^2}.
\end{aligned}
$$

Using this relation:

$$
\begin{aligned}
\varphi_R = dm/dR &= 1/\{\partial(u_m/u_c)/\partial m\} \\
\varphi_c = dm/dc &= -\{\partial(u_m/u_c)/\partial c\}/\{\partial(u_m/u_c)/\partial m\} = m/c
\end{aligned}
$$

we obtain:

$$
\begin{aligned}
\eta &= -R/\{m\partial(u_m/u_c)/\partial m\} \\
&= -\frac{R\{\psi(m/c) - (m/c)\psi'(m/c)\}^2}{(m/c)\psi(m/c)\psi''(m/c)} \\
&= -\frac{\psi'(m/c)\{\psi(m/c) - (m/c)\psi'(m/c)\}}{(m/c)\psi(m/c)\psi''(m/c)} \\
\zeta &= -\{c\partial(u_m/u_c)/\partial c\}/\{m\partial(u_m/u_c)/\partial m\} = 1.
\end{aligned}
$$

Because of $\zeta = 1$,

$$\begin{aligned}\theta &= -\frac{cu_{cc}}{u_c} - \frac{mu_{cm}}{u_c} \\ &= -\frac{c\psi(m/c)w''(y)}{w'(y)}.\end{aligned}$$

Because of

$$\begin{aligned}\frac{mu_{cm}}{u_c} &= \frac{m\psi'(m/c)w''(y)}{w'(y)} - \frac{(m^2/c^2)\psi''(m/c)}{\psi(m/c) - \psi'(m/c)m/c} \\ &= -\theta\frac{m/c\psi'(m/c)}{\psi(m/c)} - \frac{(m^2/c^2)\psi''(m/c)}{\psi(m/c) - \psi'(m/c)m/c},\end{aligned}$$

we obtain

$$\begin{aligned}\eta\frac{mu_{cm}}{u_c} &= \theta\frac{m/c\psi'(m/c)}{\psi(m/c)}\frac{\psi'(m/c)\{\psi(m/c) - (m/c)\psi'(m/c)\}}{(m/c)\psi(m/c)\psi''(m/c)} \\ &\quad + \frac{(m/c)\psi'(m/c)}{\psi(m/c)} \\ &= \frac{(m/c)\psi'(m/c)}{\psi(m/c)}(1-\theta\eta).\end{aligned}$$

For example, we consider

$$\psi(z) = [(1-\alpha) + \alpha z^{\frac{\eta-1}{\eta}}]^{\frac{\eta}{1-\eta}}.$$

Then

$$\psi'(z) = \alpha z^{\frac{\eta-1}{\eta}-1}[(1-\alpha) + \alpha z^{\frac{\eta-1}{\eta}}]^{\frac{\eta}{\eta-1}-1} > 0,$$

$$\frac{z\psi'}{\psi} = \frac{\alpha z^{\frac{\eta-1}{\eta}}}{(1-\alpha) + \alpha z^{\frac{\eta-1}{\eta}}}.$$

The money function is derived from

$$\begin{aligned}R &= \frac{\psi'}{\psi - z\psi'} \\ &= \frac{\alpha z^{\frac{\eta-1}{\eta}-1}[(1-\alpha) + \alpha z^{\frac{\eta-1}{\eta}}]^{\frac{\eta}{\eta-1}-1}}{[(1-\alpha) + \alpha z^{\frac{\eta-1}{\eta}}]^{\frac{\eta}{\eta-1}} - \alpha z^{\frac{\eta-1}{\eta}}[(1-\alpha) + \alpha z^{\frac{\eta-1}{\eta}}]^{\frac{\eta}{\eta-1}-1}}\end{aligned}$$

$$
\begin{aligned}
&= \frac{\alpha z^{\frac{-1}{\eta}}}{1-\alpha+\alpha z^{\frac{\eta-1}{\eta}}-\alpha z^{\frac{\eta-1}{\eta}}} \\
&= \frac{\alpha z^{-\frac{1}{\eta}}}{1-\alpha}.
\end{aligned}
$$

Therefore, $z = ((1-\alpha)R/\alpha)^{-\eta}$. Substituting $z = ((1-\alpha)R/\alpha)^{-\eta}$ into $z\psi'/\psi$ leads to:

$$
\begin{aligned}
\frac{z\psi'}{\psi} &= \frac{\alpha((1-\alpha)R/\alpha)^{1-\eta}}{(1-\alpha)+\alpha((1-\alpha)R/\alpha)^{1-\eta}} \\
&= \frac{((1-\alpha)/\alpha)^{-1}((1-\alpha)R/\alpha)^{1-\eta}}{1+((1-\alpha)/\alpha)^{-1}((1-\alpha)R/\alpha)^{1-\eta}} \\
&= \frac{((1-\alpha)/\alpha)^{-\eta}R^{1-\eta}}{1+((1-\alpha)/\alpha)^{-\eta}R^{1-\eta}} \\
&= \frac{R^{1-\eta}}{((1-\alpha)/\alpha)^{\eta}+R^{1-\eta}}.
\end{aligned}
$$

Therefore,

$$
\eta\frac{mu_{cm}}{u_c} = \left(\frac{(1-\alpha)^{-\eta}R^{1-\eta}}{\alpha^{-\eta}+(1-\alpha)^{-\eta}R^{1-\eta}}\right)(1-\theta\eta).
$$