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Introduction and Summary

This paper intends to analyze the Keynesian income multiplier in the framework of the input-output model. By such an analysis, we consider whether the Keynesian macroeconomic income multiplier can be determined or not in the case in which the whole industry is disaggregated into n different industries and in which interindustry input requirements are taken into consideration.

It will be shown that, in such an analysis, if final demand for any good is determined as a function of the level of national income, the macroeconomic income multiplier not only can be determined but also takes exactly the same form, that is, $1/(1-c)$, as in the one-commodity macroeconomic model (Section 2).

Based on the newly defined concepts of the interindustry income multiplier and the interindustry final demand multiplier (Section 3), we will consider in Section 4 the disaggregations of the macroeconomic income multiplier into the 'industrywise' income multipliers and into the industrywise final demand multipliers. It will then be shown that both the industrywise income and final demand multipliers cannot be determined as ratios independent of the proportions of the components (dI_j, $j=1,2,...,n$) of dI. However it will be clarified that the *income-induced* final demand multiplier is determined as a ratio independent of the proportions of dI_j's.

In Section 5, we consider the general case in which the propensities to spend for each good are not necessarily equal for all industries. In this general case the macroeconomic income multiplier cannot necessarily be determined. However, if we assume that the marginal propensities to spend for each good are equal for all industries, the macroeconomic income multiplier can be determined, taking the same form $1/(1-c)$, and all the results in the previous sections hold true in this case.

1. *The Model*

Assume that an economy has n produced goods and m factors of production. For each $j=1,2,\ldots,n$, let us denote by $a_{ij}\geqq 0$ the minimum quantities of inputs of produced goods $(i=1,2,\ldots,n)$ necessary to secure one unit of the j th goods and by $v_{ij}\geqq 0$ the value added by the factors of production $(i=1,2,\ldots,m)$ per unit of the j th good. Then the total value added per unit of the j th good, denoted by v_j, equals $\sum_{i=1}^{m} v_{ij}$. We assume that a_{ij} and v_j are constant for all i and j, that $v_j>0$ for all j, and that the input-output matrix $A=[a_{ij}]$ is indecomposable and fulfills the Hawkins-Simon condition.

Let $p_i(i=1,2,\ldots,n)$ denote the price of the i th good. In equilibrium price equals unit cost for each industry, so that we have

$$p_j=a_{1j}p_1+a_{2j}p_2+\cdots+a_{nj}p_n+v_j \qquad (1)$$

Let $x_i(i=1,2,\ldots,n)$ denote the output of the i th industry. Then $\sum_{i=1}^{n} v_i x_i \equiv Y$ signifies the total value added of the economy as a whole. Let us call Y *national income*.

Assume that real final demand for the i th good consists of two parts: one is the part of the final demand which is independent of the level of national income Y and the other is the part of the final demand which depends on the level of Y. The former, denoted by b_i, is called *autonomous* demand and the latter, denoted by f_i, *income-induced* demand. Let F_i denote the value of the income-induced final demand for the i th good, that is, $F_i \equiv p_i f_i$. It is assumed that f_i is a function of Y and vector $p\equiv(p_1,p_2,\ldots,p_n)'$ (where the prime means transposition), so that $f_i = f_i(Y,p)$. We assume that, given p, $f_i(i=1,2,\ldots,n)$ are non-decreasing functions of Y, and that $f_i(0,p)=0$, that is, all the income-induced final demand functions equal zero if national income is zero. The functions $f_i(Y,p)$ $(i=1,2,\ldots,n)$ are given and assumed differentiable with respect to Y and that $1 > \sum_{i=1}^{n}(\partial F_i/\partial Y) > 0$.

If we denote by $y_i(i=1,2,\ldots,n)$ the total real final demand for the ith good, we have

$$y_i = b_i + f_i(Y,p) \qquad (i=1,2,\ldots,n) \tag{2}$$

In equilibrium the output of the ith industry, x_i, is absorbed by the interindustry input requirements and the final demand, so that we have

$$x_i = a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n + y_i \tag{3}$$

for $i=1,2,\ldots,n$.

Finally it may be convenient here to summarize which magnitudes are assumed to be given parameters and which to be variables. The following are parameters: a_{ij}, v_i, and b_i for all $i, j=1,2,\ldots,n$. The following are variables determined from the above parameters and the equations (1)–(3): x_i, y_i and p_i for all i, that is, $x_i = x_i(A, v, b)$, $y_i = y_i(A, v, b)$ and $p_i = p_i(A, v, b)$ for all i, where $v \equiv (v_1, v_2, \ldots, v_n)'$ and $b \equiv (b_1, b_2, \ldots b_n)'$.

In the following sections, we do comparative statics to analyze how the above variables and some functions of them are influenced by changes in the parameters b_i, $i=1,2,\ldots,n$, and by changes in functions of them, that is, I_i and I. Meanwhile A and v are considered to be constant.

2. *The Macroeconomic Income Multiplier*

Let I_i denote the value of the autonomous demand for the ith good, so that $I_i \equiv p_i b_i$ for $i=1,2,\ldots,n$. Let us call $I \equiv \sum_{i=1}^{n} I_i$ *aggregate autonomous demand.* We define *the macroeconomic income multiplier* as the ratio of the change in the equilibrium national income

(Y) caused by a change in the aggregate autonomous demand (I) to the change in I, that is, as dY/dI.

Theorem 1 $dY/dI=1/(1-c)$, where $c\equiv\sum_{i=1}^{n}(\partial F_i/\partial Y)$.

Proof In order to use matrix terminology, let us define $x\equiv(x_1, x_2, \ldots, x_n)'$, $y\equiv(y_1, y_2, \ldots, y_n)'$, $F=(F_1, F_2, \ldots, F_n)'$, $g\equiv(\partial F_1/\partial Y, \partial F_2/\partial Y, \ldots, \partial F_n/\partial Y)'$, $P\equiv\text{diag }(p_1, p_2, \ldots, p_n)$, $e\equiv(1, 1, \ldots, 1)'$, and $E\equiv\text{diag }(1, 1, \ldots, 1)$, i.e., unit matrix.

Eq. (3) may be rewritten as $x=Ax+y$...............(4), so that $y=[E-A]x$. Hence $p'y=p'[E-A]x$. Eq. (1) may be rewritten as $p'=p'A+v'$, so that $v'=p'[E-A]$...............(5). Thus we have $p'y=v'x$, that is, $p'y=Y$...............(6), which means that the aggregate final demand equals national income.

By assumption A is constant, and by Eq. (5), we have $p'=v'[E-A]^{-1}$...............(7), where it must be remarked that existence of matrix $[E-A]^{-1}$ is ensured by the assumed Hawkins-Simon condition. Moreover, positivity of all p_i's is ensured by positivity of all v_i's and the assumed indecomposability of A. Since v is constant, it follows that p is constant, so that we may put $f_i(Y, p)=f_i(Y)$ and we can denote $F_i\equiv p_i f_i(Y)\equiv F_i(Y)$.

Since p is a strictly positive vector, Eq. (2) may be rewritten as $y=b+P^{-1}F(Y)$...............(8), so that $p'y=p'b+p'P^{-1}F(Y)=p'b+e'F(Y)$. By Eq. (6), we have $Y=p'b+e'F(Y)$. Let us note here that $I=p'b$ and define a function $S(Y)$ by $S(Y)\equiv Y-e'F(Y)$. Then we have $Y=I+e'F(Y)$ or $S(Y)=I$. It follows that $S'(Y)dY=dI$, so that $dY/dI=S'(Y)$. Now $S'(Y)=(Y-e'F(Y))'=1-e'g=1-c$. (Q. E. D.)

Theorem 1 implies that the change dY in national income caused by a change db in the autonomous demand vector is independent of the relative proportions of the components db_i, $i=1, 2, \ldots, n$, of db, and depends only on the change dI of the aggregate value I of b.

3. *The Interindustry Income and Final Demand Multipliers*

The following theorem is fundamental for later discussions.

Theorem 2 $dy=\left[E+\left(\frac{1}{1-c}\right)P^{-1}gp'\right]db$

Proof By Eq. (8), we have $dy=db+d(P^{-1}F(Y))=db+P^{-1}gdY$. By Theorem 1, $dY=\left(\frac{1}{1-c}\right)dI$, so that we have $dy=db+P^{-1}g\left(\frac{1}{1-c}\right)dI$. Since p is constant, $dI=p'db$. Thus we have $dy=db+P^{-1}g\left(\frac{1}{1-c}\right)p'db=\left[E+\left(\frac{1}{1-c}\right)P^{-1}gp'\right]db$. (Q. E. D.)

For all $i, j=1, 2,..., n$, we define *the interindustry income multiplier* h_{ij} as the ratio of the change in the total value added of the ith industry caused by a change in the value of the autonomous demand for the jth good (I_j) to the change in I_j. In symbol, $h_{ij}\equiv\partial(v_ix_i)/\partial I_j$.

Theorem 3 $H\equiv[h_{ij}]=V[E-A]^{-1}P^{-1}\left[E+\left(\frac{1}{1-c}\right)ge'\right]$, where $V\equiv$ diag $(v_1, v_2,..., v_n)$.

Proof Put $u\equiv(u_1, u_2,..., u_n)'$ and $u_i\equiv v_ix_i$, $i=1, 2,..., n$. Then $u=Vx$ so that $du=d(Vx)=Vdx=V[E-A]^{-1}dy$ (from Eq. (4)) $=V[E-A]^{-1}\left[E+\left(\frac{1}{1-c}\right)P^{-1}gp'\right]db$ (by Theorem 2) $=V[E-A]^{-1}\left[E+\left(\frac{1}{1-c}\right)P^{-1}gp'\right]P^{-1}d(Pb)=V[E-A]^{-1}\left[P^{-1}+\left(\frac{1}{1-c}\right)P^{-1}ge'\right]d(Pb)$ $=V[E-A]^{-1}P^{-1}\left[E+\left(\frac{1}{1-c}\right)ge'\right]d(Pb)$. Noting $Pb=(I_1, I_2,..., I_n)'$, this proves Theorem 3. (Q. E. D.)

For all $i, j=1, 2,..., n$, we define *the interindustry final demand multiplier* z_{ij} as the ratio of the change in the total value of the final demand for the ith good caused by a change in the value of the autonomous demand for the jth good (I_j) to the change in I_j. In symbol, $z_{ij}\equiv\partial(p_iy_i)/\partial I_j$.

Theorem 4 $Z\equiv[z_{ij}]=E+\left(\frac{1}{1-c}\right)ge'$.

Proof Put $t\equiv(t_1, t_2,..., t_n)'$ and $t_i\equiv p_iy_i$, $i=1, 2,..., n$. Then $t=Py$

so that $dt = d(Py) = Pdy = P\left[E + \left(\frac{1}{1-c}\right)P^{-1}gp'\right]db$ (by Theorem 2) $= \left[P + \left(\frac{1}{1-c}\right)gp'\right]db = \left[P + \left(\frac{1}{1-c}\right)gp'\right]P^{-1}\,d(Pb) = \left[E + \left(\frac{1}{1-c}\right)g\right.$ $\left.e'\right]d(Pb)$. (Q. E. D.)

Theorem 5 If the n industries are independent of each other so that any industry does not require any inputs from the other industries, then the interindustry income multiplier and the interindustry final demand multiplier are equal.

Proof Put $A = O$ in Theorem 3. Then, by Eq. (5), $v = p$ so that $V = P$. Hence $H = E + \left(\frac{1}{1-c}\right)ge' = Z$. (Q. E. D.)

4. *The Industrywise Income, Final Demand, and Income-Induced Final Demand Multipliers*

For all $i = 1, 2, \ldots, n$, we define *the industrywise income multiplier* r_i as the ratio of the change in the total value added of the ith industry caused by a change in the aggregate autonomous demand (I) to the change in I, that is, $r_i \equiv d(v_i x_i)/dI$. A change in I, or dI, consists of dI_j, $j = 1, 2, \ldots, n$, and by definition $h_{ij} = \partial(v_i x_i)/\partial I_j$. Hence we have $r_i = \sum_{j=1}^{n}(\partial(v_i x_i)/\partial I_j)(dI_j/dI) = \sum_{j=1}^{n} h_{ij}(dI_j/dI)$...............(9). It is clear by this equation and by the explicit form of H in Theorem 3 that r_i depends on the ratios dI_j/dI, $j = 1, 2, \ldots, n$, that is, on the proportions which dI_j's occupy in dI. Thus we have the following

Theorem 7 The indusrywise income multiplier cannot be determined as a ratio independent of the proportions of dI_j, $j = 1, 2, \ldots, n$, in dI.

For all $i = 1, 2, \ldots, n$, we define *the industrywise final demand multiplier* q_i as the ratio of the change in the total value of the final demand for the ith good caused by a change in the aggregate autonomous demand (I) to the change in I, that is, $q_i \equiv d(p_i y_i)/dI$.

By definition $z_{ij} = \partial(p_i y_j)/\partial I_j$, and, by Theorem 4, $z_{ij} = \delta_{ij} +$

$\left(\frac{1}{1-c}\right)F_i'$, where $\delta_{ij}=1$ if $i=j$ and 0 if $i\neq j$. Hence we have $q_i=$ $=\sum_{j=1}^n(\partial(p_iy_i)/\partial I_j)(dI_j/dI)=\sum_{j=1}^n z_{ij}(dI_j/dI)=\sum_{j=1}^n\left(\delta_{ij}+\left(\frac{1}{1-c}\right)F_i'\right)$ $(dI_j/dI)=(dI_i/dI)+\sum_{j=1}^n\left(\frac{1}{1-c}\right)F_i'(dI_j/dI)=(dI_i/dI)+\left(\frac{1}{1-c}\right)F_i'$ $\sum_{j=1}^n(dI_j/dI)=(dI_i/dI)+\left(\frac{1}{1-c}\right)F_i'$..............(10).

Theorem 8 $q_i=(dI_i/dI)+\left(\frac{1}{1-c}\right)F_i'$, $i=1,2,\ldots,n$. The industry-wise final demand multiplier cannot be determined as a ratio independent of the proportion of dI_i in dI, but its part corresponding to the change in the value of the income-induced final demand for the ith good is determined independently of dI_i/dI.

Proof The proof of the first part of this theorem is obvious. Now for the proof of the last part, it will be sufficient to show that $\partial(p_if_i)/\partial I=\left(\frac{1}{1-c}\right)F_i'$..............(11), where p_if_i is the value of the income-induced final demand for the ith good. $\partial(p_if_i)/\partial I=\partial(p_i(y_i-b_i))/\partial I=\partial(p_iy_i-I_i)/\partial I=(\partial(p_iy_i)/\partial I)-(\partial I_i/\partial I)=q_i-(\partial I_i/\partial I)=\left(\frac{1}{1-c}\right)F_i'$, which in independent of dI_i. (Q. E. D.)

Let us call $\partial(p_if_i)/\partial I$ *the income-induced final demand multiplier.*

5. *Industrywise Different Propensities to Spend: the General Case*

All the above results are based on the assumption that the income-induced final demand for any good is determined by the level of national income Y. This assumption may be relaxed as follows: let f_{ij} denote the income-induced final demand for the ith good from the total value added for the jth industry (v_jx_j), and suppose f_{ij} is a non-decreasing function of v_jx_j, being differentiable with respect to v_jx_j. Let us call p_if_{ij} *the industrywise propensity to spend* for the ith good. Then we have

$$y=b+G(x)e \qquad (12)$$

where $G=[f_{ij}]$. Clearly in this general case we cannot proceed so simply as in the proof of Theorem 1. However the interindustry and industrywise multipliers h_{ij}, z_{ij}, r_i, and q_i are determined in this case. Indeed, by Eq. (12), we have $dy=db+d(G(x)e)$ and $d(G(x)e)=[d(\sum_{j=1}^{n}f_{ij}(v_jx_j))]=[\sum_{j=1}^{n}f'_{ij}v_jdx_j]=UVdx$, where $U\equiv[f'_{ij}]$. Hence, from Eq. (4), we have $dx=Adx+db+UVdx$, so that we have

$$dx=[E-A-UV]^{-1}db \tag{13}$$

provided that the inverse matrix on the right-hand side exists. Therefore $dy=[E-A]dx=[E-A][E-A-UV]^{-1}db=\{[E-A]^{-1}\}^{-1}[E-A-UV]^{-1}db=\{[E-A-UV][E-A]^{-1}\}^{-1}db=\{E-UV[E-A]^{-1}\}^{-1}db$..............(14), so that if denote $B\equiv UV[E-A]^{-1}$, we have

$$dy=[E-B]^{-1}db=(E+\textstyle\sum_{w=0}^{\infty}B^w)db \tag{15}$$

provided that the sequence $E+B+B^2+\ldots$ is convergent. The formulae for the interindustry and industrywise multipliers can be easily derived from Eq. (14) or (15). However the macroeconomic income multiplier cannot be determined independently of dI_i's in this general case because the effect of dI on Y generally differs depending on the proportions of the components dI_i, $i=1,2,\ldots,n$, of dI.

Let us therefore consider under what conditions the macroeconomic income multiplier can be determined.

Theorem 9 If, for all $i=1,2,\ldots,n$, the industrywise marginal propensities to spend for the i th good are equal for all industries, so that $f'_{i1}=f'_{i2}=\ldots=f'_{in}$, and if $0<\sum_{i=1}^{n}p_if'_{ij}<1$, then we can put $\sum_{i=1}^{n}p_i f'_{ij}=c$ and Theorems 1-8 hold true.

Proof If we put $f'_{ij}\equiv\tilde{f}_i'$ and $\tilde{f}\equiv(\tilde{f}_1',\tilde{f}_2',\ldots,\tilde{f}_n')'$, we have $df_i=d(\sum_{j=1}^{n}f_{ij})=\sum_{j=1}^{n}df_{ij}=\sum_{j=1}^{n}(f'_{ij}d(v_jx_j))=\sum_{j=1}^{n}(\tilde{f}_i'd(v_jx_j))=\tilde{f}_i'\sum_{j=1}^{n}d(v_jx_j)=\tilde{f}_i'dY$..............(16). Hence we have $df_i/dY=\tilde{f}_i'$ so that $\tilde{f}_i'=f_i'$, $i=1,2,\ldots,n$. Since $F_i\equiv p_if_i$ and p_i is constant, we have $\partial F_i/\partial Y=\partial(p_if_i)/\partial Y=p_i(\partial f_i/\partial Y)=p_i\tilde{f}_i'$. Therefore by the definition of c in Theorem 1, we have $c=\sum_{i=1}^{n}p_i\tilde{f}_i'=\sum_{i=1}^{n}p_if'_{ij}$. This proves the first part of Theorem 9.

By Eq. (12), we have $p'y=p'b+p'G(x)e$, that is, $Y=I+p'G(x)e$, so that we have $dY=dI+d(p'G(x)e)=dI+p'd(G(x)e)$. And we have $d(G(x)e)=UVdx$ as was shown just before Eq. (13). Since $U\equiv[f'_{ij}]$, we have $U=\tilde{f}e'$, so that $d(G(x)e)=\tilde{f}e'Vdx=\tilde{f}e'V[E-A]^{-1}$ $dy=\tilde{f}v'[E-A]^{-1}dy=\tilde{f}p'dy=\tilde{f}d(p'y)=\tilde{f}dY$...............(17). Hence we have $dY=dI+p'\tilde{f}dY=dI+c(dY)$, so that $dY=\left(\frac{1}{1-c}\right)dI$. This proves Theorem 1.

$dy=db+d(G(x)e)=db+\tilde{f}dY$ (by Eq. (17))$=db+\left(\frac{1}{1-c}\right)\tilde{f}dI=$ $db+\left(\frac{1}{1-c}\right)\tilde{f}p'db=\left[E+\left(\frac{1}{1-c}\right)\tilde{f}p'\right]db$............(18), which, in view of $\tilde{f}=P^{-1}g$, proves Theorem 2. Based on Theorems 1 and 2, the proofs of Theorems 3-8 are similar to those in the previous sections. (Q. E. D.)

Conclusions

It became clear that, if the industrywise marginal propensities to spend for each good are equal for all industries, the concept of the macroeconomic income multiplier can be incorporated into the input-output model. This implies that, under the same condition, the effect of the economic policy to change the autonomous final demand vector on the level of national income (apart from how to finance the change in public expenditure) is independent of the proportions of the components of the change in the autonomous final demand vector, depending only on the total amount of the change in public expenditure.

From the microeconomic viewpoint, the industrywise effects of the fiscal policy come to be considered. The effect of the policy on the allocation of factors of production among industries varies depending on the proportions of the components of the policy. However, under the above condition, the effect of the policy on the income-induced final demand for each good is independent of the proportions of the components of the policy, since the change in the level of national income caused by the policy is independent of those proportions. It depends only on the total change in public expenditure. In other words, even if the proportions of the components of the policy are different, it does not alter the ***repercussions*** (as distinct from the ***direct effect***) of the policy on the final demand for each good, as long as the total change in public expenditure is constant.

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