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Koichi MIYAZAKI

Introduction

From the mathematical viewpoint it seems to be the best here to refer to the well-known mathematical formula[1)]

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n=e. \quad \text{.........................(1)}$$

It may be worth while to interpret Eq. (1) *in terms of the compound interest.* In short, Eq. (1) can be said to mean that, if time is infinitesimally divisible so that the strictly 'instantaneous' rate of interest can be considered, and is assumed to be constantly just 100% a year, the total compound rate of interest equals $e-1 \fallingdotseq 2.718-1=171.8\%$ per year.

It is one of the aims of this paper to generalize Eq. (1) to the following equation

$$\lim_{n\to\infty}\prod_{\nu=1}^{n}\left(1+\int_{\frac{\nu-1}{n}}^{\frac{\nu}{n}} b(t)dt\right)=e^{\int_0^1 b(t)dt}, \quad \text{...............(2)}$$

where[2)] $b(t)$ denotes the (not necessarily constant) instantaneous rate of interest defined on $0\leqq t\leqq 1$. (If we put $b(t)\equiv 1$, Eq. (2) will reduce to Eq. (1) above.) The term on the right-hand side of Eq. (2) minus unity may be called as *the exponential compound*

1) Eq. (1) is said to be the first definition in the history of mathematics of the base e of the exponential function e^x [4, pp. 9, 189].

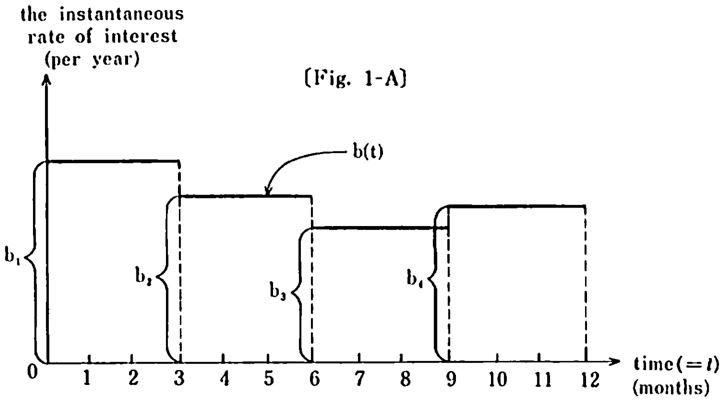
2) A formula which is similar to, but not quite the same as, Eq. (2) has been already known. See, e. g., [3, p. 100].

interest.

The instantaneous rate of interest and its time-profile function $b(t)$ will be explained in Section 1 of this paper. In Section 2, the simple-sum interest is defined, and the relation between the simple-sum interest and the compound interest will be illustrated in the case of discretely divided periods. The compound interest in the divided-time case gives an interpretation to the product on the left-hand side of Eq. (2) above. In Section 3, Eq. (2) will be proved (Theorem 1). In Section 4, the new concept *'the i-th interest stratum'* will be introduced, and its explicit form will be derived (Theorem 2). In Section 5, the product on the left-hand side of Eq. (2) will be expanded, and it will be shown that the $(i+1)$-th term of the expanded series converges to the term $\left(\int_0^1 b(t)dt\right)^i/i!$ for all $i=1,2,...$(Theorem 3), a result which virtually clarifies, so to say, *how* Eq. (2) holds. And Section 6 will define *the k-stratum interest.* A double-series convergence property of the k-stratum interest to the exponential compound interest will be clarified (Theorem 4). Conclusion summarizes the main results obtained in this paper.

1. The Instantaneous Rate of Interest

The most fundamental concept of this paper is the 'instantaneous' rate of interest. Let us therefore consider it in this section. An analogous usage of the word 'instantaneous' may be easily found, e.g., in the term, "maximum instantaneous wind speed." The wind speed w meters per second will equal $w\times60\times60$ meters $=3.6\,w$ kilometers per hour. The instantaneous rate of interest is just like the $3.6\,w$ km in the analogy (as distinct from the w meters). It signifies how much the interest will amount to *in one year* if the unit principal were to continue to produce interest for a year at the *same* (i.e., constant) pace *as* at the moment in question. Remark that the validity of the analogy is limited, since interest can be *plowed back* to the principal (see Section 2, below). Suppose that the instantaneous rate of interest is constant at the level b_1 all through the year, and that any interest is not plowed back to the initial unit principal. Then the total interest obtained till the end of the year equals b_1. If as Fig. 1-A shows the rate of interest is not constant through the year but is changed from b_1 to b_2, b_2 to b_3, and b_3 to b_4, at the end of March, June, and September, respectively, the part of the interest obtained only in the period from January through March, e. g., will equal $b_1/4$, that is, one fourth of the yearly interest b_1 in the above case where the



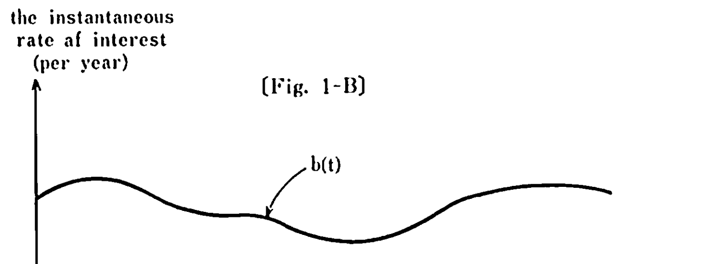


Fig. 1
(Examples of the Time Profile of the Instantaneous Rate of Interest)

rate is constant all through the year, provided that there is not any plowing back. It represents the simple linear (or constant-paced) growth of the interest for $0 \leqq t \leqq 1/4$. In this case, the total interest obtained till the end of the year (without any plowing back) will equal to $(b_1/4)+(b_2/4)+(b_3/4)+(b_4/4)$.

The time-schedule of the rate of interest may take various forms. If another example is taken in which the rate changes much more gradually, the curve may be like Fig. 1-B.

The time-schedule of the instantaneous rate of interest is written as $b(t)$, $0 \leqq t \leqq t_0$, where t_0 is a positive constant, and assumed to be one year (say) throughout this paper, or $t_0=1$. As for the properties of $b(t)$, we place the following

Assumption $b(t)$ is Riemann-integrable, bounded from above, and has a positive lower bound.

2. *The 'Simple-sum' Interest and its Relation to the Compound Interest*

Mathematically, *the simple-sum rate of interest* for the period $t_1 \leqq t \leqq t_2$ is defined by

$$\int_{t_1}^{t_2} b(t)dt.$$

Analogically, it represents the total distance of simply how far 'the wind' proceeds from time t_1 through t_2. In case of Fig. 1-A, if the wealth-holder lends the principal through one year and withdraws *all* the interests obtained at the end of March, June, September, and December *without any plowing back throughout the year*, then the total of those interests equals $(b_1/4)+(b_2/4)+(b_3/4)+(b_4/4)$, or the simple-sum interest for the year. But if he lends the money and *plows back all* the interests obtained at the end of March, June, and September, then to how much will amount the total of the interests, including interests on interests, interests on interests on interests, etc.? The answer to this question can obviously be written as

$$(1+m_1)(1+m_2)(1+m_3)(1+m_4)-1$$
$$=m_1+m_2+m_3+m_2\,m_3+m_3\,m_1+m_1\,m_2$$
$$+m_2\,m_3\,m_4+m_1\,m_3\,m_4+m_1\,m_2\,m_4+m_1\,m_2\,m_3$$
$$+m_1\,m_2\,m_3\,m_4, \quad \text{.......................................}(3)$$

where $m_i=b_i/4$ $(i=1,2,3,4)$.

But if he tries to maximize the final sum of money at the end of the year, he will at least think of plowing back interests more frequently than three times in the example, in order to end up the year having a greater sum.

As a quite natural consequence of our definition of the instantaneous rate of interest over time, it might be taken for granted that he is allowed to plow back at *any* moment of the year, so that he will plow back as frequently as he can, since it is assumed in this paper that the total cost of plowing back is assumed negligible. However if the year is divided into periods, it is assumed that he is not allowed to plow back at any other moments than at the ends of the periods.

3. A Limit Representation of the Exponential Compound Interest

The first of our main results of this paper is the following

Theorem 1 If a year can be divided into as many time-segments of equal length (i.e., as large n) as is desired, the compound interest which is obtained can be in as small a neighborhood as is liked of, and indeed it converges to, $\exp\left(\int_0^1 b(t)dt\right)$, though it cannot exceed that amount.

Proof Let us denote by D_n the case in which the year is divided into n segments of the same length, $1/n$ year. Then, the unit principal and the compound interest in D_n will be written as

$$\prod_{\nu=1}^{n}(1+m_{n\nu}), \text{ where } m_{n\nu}=\int_{(\nu-1)/n}^{\nu/n} b(t)dt.$$

(See the example for the case $n=4$ in Section 2 above.) Since

$$\int_0^1 b(t)dt = \sum_{\nu=1}^{n} m_{n\nu},$$

we may write

$$1<\alpha_n \equiv \frac{\prod_{\nu=1}^{n} \exp m_{n\nu}}{\prod_{\nu=1}^{n}(1+m_{n\nu})} = \prod_{\nu=1}^{n}\left[\frac{1+m_{n\nu}+\frac{m_{n\nu}^2}{2!}+\frac{m_{n\nu}^3}{3!}+\cdots}{1+m_{n\nu}}\right]$$

$$= \prod_{\nu=1}^{n}\left[1+\frac{\frac{m_{n\nu}^2}{2!}+\frac{m_{n\nu}^3}{3!}+\frac{m_{n\nu}^4}{4!}+\cdots}{1+m_{n\nu}}\right]. \qquad \text{...............(4)}$$

Since $b_2 \geqq b(t)>0$ for all t, $0 \leqq t \leqq 1$, we have $b_2/n \geqq m_{n\nu}$ for all $\nu=1, 2, \cdots, n$, $n=1, 2, \cdots$. Let us rewrite b_2 as b for simplicity. Then, we have

$$\alpha_n \leqq \prod_{\nu=1}^{n}\left(1+\frac{(b/n)^2}{2!}+\frac{(b/n)^3}{3!}+\frac{(b/n)^4}{4!}+\cdots\right)$$

$$=\left(1+\frac{(b/n)^2}{2!}+\frac{(b/n)^3}{3!}+\frac{(b/n)^4}{4!}+\cdots\right)^n. \qquad \text{.........(5)}$$

The right-hand side of this Eq. (5) *minus* unity will equal

$$\left(1+\frac{(b/n)^2}{2!}+\frac{(b/n)^3}{3!}+\frac{(b/n)^4}{4!}+\cdots\right)^n-1$$

$$=\left(\frac{(b/n)^2}{2!}+\frac{(b/n)^3}{3!}+\frac{(b/n)^4}{4!}+\cdots\right)\cdot$$

$$\left[\sum_{j=0}^{n-1}\left(1+\frac{(b/n)^2}{2!}+\frac{(b/n)^3}{3!}+\cdots\right)^j\right]$$

$$\leqq\left(\frac{b^2/n}{2!}+\frac{b^3/n^2}{3!}+\frac{b^4/n^3}{4!}+\cdots\right)\cdot$$

$$\left(1+\frac{(b/n)^2}{2!}+\frac{(b/n)^3}{3!}+\cdots\right)^{n-1}. \qquad \text{..................(6)}$$

But

$$\frac{(b/n)^2}{2!}+\frac{(b/n)^3}{3!}+\frac{(b/n)^4}{4!}+\cdots$$

$$\leqq (b/n)^2\left(1+(b/n)+\frac{(b/n)^2}{2!}+\frac{(b/n)^3}{3!}+\cdots\right)$$

$$\leqq (b/n)^2 \exp(b/n). \qquad \text{(7)}$$

For all large enough n such that $\exp(b/n)<2$ and $b^2/n<1$, we have

$$1+(b/n)^2\exp(b/n)<1+(2/n),$$

so that

$$\lim_{n\to\infty}(1+(b/n)^2\exp(b/n))^{n-1}\leqq\lim_{n\to\infty}(1+(2/n))^{n-1}$$
$$=\exp 2. \qquad \text{(8)}$$

Hence, by (4), (5), (6), (7), and (8),

$$0\leqq\lim_{n\to\infty}(\alpha_n-1)$$
$$\leqq\lim_{n\to\infty}((b^2/n)\exp(b/n))\exp 2=0,$$

that is, $\lim_{n\to\infty}\alpha_n=1$. (Q. E. D.)

Thus we have proved Eq. (2) in Introduction. $\exp\left(\int_0^1 b(t)dt\right)$ is the solution of the differential equation $df/dt=b(t)f(t)$ for $f(0)=1$, where $f(t)$ denotes the amount of the principal at time t, $0\leqq t\leqq 1$, and $b(t)$ the rate of increase of the principal. Therefore, Theorem 1 can be interpreted to mean that the solution of $df/dt=b(t)f(t)$ $(f(0)=1)$ is the desired upper bound of the wealth-holder's total of principal and interest for D_n at the end of the year: namely that his total for D_n is less than that if n is finite, but that it tends to that as n goes to infinity.

4. *The Interest Strata*

Let us first define

$$R_{n0}(s)=1, \quad (s=0,1,2,\cdots,n)$$
$$R_{ni}(s)=\sum_{r=i}^{s} m_{nr}R_{n,i-1}(r-1), \qquad \text{(9)}$$
$$(i=1,2,\cdots,s,\ s=1,2,\ldots,n)$$

and $R_{ni}(0)=0$, $i=1,2,\cdots,n$ for all $1\leqq n<+\infty$. $R_{n0}(s)=1$, $s=0,1,2,\cdots,n$ means that the total of the principal and interest is constant if no direct interests are plowed back to the principal all through the periods $s=1,2,\cdots,n$. $R_{n1}(s)$, e.g., signifies the total of all the *direct* interests produced on the unit principal through the periods $r=1,2,\cdots,s$. Generally $R_{ni}(s)$ signifies the total of all the direct interests produced on $R_{n,i-1}(r)$ of the periods $r=i-1,i,\cdots,s-1$, through the periods i, $i+1$, ..., s.

Let us rewrite $R_{ni}(n)$ as simply R_{ni}, and call it as *the i-th interest stratum for D_n*. Then we have the following

Theorem 2

$$R_{ni}=\sum_{c\in C_{H_n,i}}\left(\prod_{j=1}^{i} m_{nc_j}\right), \quad \cdots\cdots\cdots\cdots\cdots\cdots(10)$$

where $H_n\equiv\{1,2,\cdots,n\}$, and $C_{H_n,i}$ denotes the set of all combinations each of which consists of i different elements taken out of the set H_n. The symbol c can be written as $(c_1,c_2,\cdots,c_i)$, where $i\geqq c_1>c_2>\cdots>c_i\geqq 1$, $i=1,2,\ldots,n$.

Proof In this proof we put $m_{nr}=m_r$ and $R_{ni}(u)=R_i(u)$ for brevity. It is easy to verify

$$C_{H_u,i}=\bigcup_{k=i}^{u}\{(k,y):y\in C_{H_{k-1},i-1}\},\quad i=2,3,\cdots,u, \cdots\cdots\cdots(11)$$

where it may be noted that the $u-i+1$ sets on the right-hand side of (11) are disjoint each other. It is well known that $\prod_{i=1}^{u}(x+m_i)=x^u+\sum_{i=1}^{u}x^{u-i}A_i(u)$, where

$$A_i(u)=\sum_{c\in C_{H_u,i}}\prod_{j=1}^{i} m_{c_j}. \cdots\cdots\cdots\cdots\cdots\cdots(12)$$

In order to prove (10), it will suffice to verify $R_i(u)=A_i(u)$ for all $i=1,2,\ldots,u$, $u=1,2,\ldots,n$. Let us first verify this equation for $i=1$, $u=1,2,\ldots,n$. By (12), we have $A_1(u)=m_1+\cdots+m_u$, and by Eq. (9), we have $R_1(u)=m_1+\cdots+m_u$. Hence $R_1(u)=A_1(u)$, for $u=1,2,\ldots,n$. Mathematical induction in the following will show that $R_i(u)=$

$A_i(u)$ holds for all $i=1,2,\cdots,u$, $u=1,2,\ldots,n$. Suppose this equation holds for $i=1,2,\ldots,u$, $u=1,2,\ldots,s-1$. Then by (9), (11) and (12) we have

$$A_i(s)=\sum_{c\in CH_{s,i}}\prod_{j=1}^{i}m_{c_j}=\sum_{u=i}^{s}\sum_{\left[\substack{c_1=u\\ c^2\in CH_{u-1,i-1}}\right]}\prod_{j=1}^{i}m_{c_j}$$

$$=\sum_{u=i}^{s}m_u\left[\sum_{c^2\in CH_{u-1,i-1}}\prod_{j=2}^{i}m_{c_j}\right]=\sum_{u=i}^{s}m_uA_{i-1}(u-1)$$

$$=\sum_{u=i}^{s}m_uR_{i-1}(u-1)=R_i(s),$$

for $i=1,2,\ldots,s$, where $c^2=(c_2,\cdots,c_i)$. Therefore $R_i(u)=A_i(u)$ holds for all $i=1,2,\ldots,u$, $u=1,2,\ldots,s$. It follows that the same equation holds for all $i=1,2,\ldots,u$, $u=1,2,\ldots,n$. Thus we have $R_i(n)=A_i(n)$. (Q. E. D.)

5. *The Convergence and the Limit of the i-th Interest Stratum for D_n as n Tends to Infinity*

By the definition of m_{nv}, Eq. (2) in Introduction can be rewritten as

$$\lim_{n\to\infty}\prod_{v=1}^{n}(1+m_{nv})=\exp\left(\int_0^1 b(t)dt\right). \quad \ldots\ldots\ldots\ldots\ldots(2')$$

So that, expanding the product $\prod_{v=1}^{n}(1+m_{nv})$ on the left-hand side, this Eq. (2′) will become

$$\begin{aligned}1+\lim_{n\to\infty}\Big[&(m_{n1}+m_{n2}+\cdots+m_{nn})\\ &+(m_{n1}\,m_{n2}+m_{n1}\,m_{n3}+\cdots+m_{n1}\,m_{nn}\\ &+m_{n2}\,m_{n3}+m_{n2}\,m_{n4}+\cdots+m_{n2}\,m_{nn}\\ &+\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ &+m_{n,n-1}\,m_{nn})\\ &+\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ &+m_{n1}\,m_{n2}\,m_{n3}\cdots m_{nn}\Big]\end{aligned}$$

$$=1+\lim_{k\to\infty}\left[\int_0^1 b(t)dt+\frac{1}{2}\left(\int_0^1 b(t)dt\right)^2+\cdots+\frac{1}{k!}\left(\int_0^1 b(t)dt\right)^k\right],$$

or, equivalently,

$$1+\lim_{n\to\infty}\left(\sum_{i=1}^{n} R_{ni}\right)=1+\lim_{k\to\infty}\sum_{i=1}^{k}\frac{\left(\int_0^1 b(t)dt\right)^i}{i!}, \quad \text{.........(13)}$$

where R_{ni} is defined by (9).

From Eq. (13), we are motivated to prove the following

Theorem 3

$$\lim_{n\to\infty} R_{ni}=\frac{\left(\int_0^1 b(t)dt\right)^i}{i!} \quad \text{........................(14)}$$

for all i, $1\leqq i<+\infty$.

As shown in Section 4, R_{ni} can be interpreted as *the i-th interest stratum for the discrete division* (D_n) *of the year into n equal-length periods,* so that Eq. (14) means that the i-th interest stratum for the division D_n converges to $\left(\int_0^1 b(t)dt\right)^i/i!$ as $n\to\infty$, where $\left(\int_0^1 b(t)dt\right)^i/i!$ can be interpreted as the i-th interest stratum in the continuous case, as is already clarified in [1].

In order to prove Theorem 3, let us first verify the following

Lemma 1 Let $\{a_{n\nu}, \nu=1,2,\cdots,n\}$ $(n=1,2,\cdots)$ denote a sequence of finite sequences consisting of positive rates of interst fulfilling the following conditions (i) and (ii) : (i) $\sum_{\nu=1}^{n} a_{n\nu}=E$ (const.) and (ii) $\lim_{n\to\infty}\left(a_{\max}^{(n)}\right)^i/\left(n\left(a_{\min}^{(n)}\right)^i\right)=0$, where $a_{\max}^{(n)}=\max_{(1\leqq\nu\leqq n)} a_{n\nu}$ and $a_{\min}^{(n)}=\min_{(1\leqq\nu\leqq n)} a_{n\nu}$. Consider the compound interest in the discrete scheme with n periods whose periodwise rates of interest equal the the terms $a_{n1},\cdots,a_{nn}$ of the n-th sequence. Then, the i-th interest stratum for such a scheme for n converges to $E^i/i!$ as $n\to\infty$.

Proof Suppose $i=1$. Then, by the definition of the 1st interest stratum in the discrete scheme, the simple sum of $a_{n1}, a_{n2}, \cdots$, and a_{nn} equals the 1st interest stratum. This proves Lemma 1 in the case of $i=1$. Suppose $i \geqq 2$. In order to prove Lemma 1, it is necessary and sufficient to show

$$\lim_{n\to\infty} \sum_{c \in CH_{n,i}} a_{nc_1} a_{nc_2} \cdots a_{nc_i} = E^i/i!, \quad \dots\dots\dots\dots\dots\dots(15)$$

since the sum on the left-hand side of (15) denotes the i-th interest stratum in the discrete case. For convenience, let us here define E_n by $E_n = \sum_{v=1}^{n} a_{nv}$, $n=1,2,\dots$. Then, $(E_n)^i$ can be expanded into

$$(E_n)^i = \sum_{g \in G} (i!/(g_1!\, g_2! \cdots g_n!)) a_{n1}^{g_1} a_{n2}^{g_2} \cdots a_{nn}^{g_n},$$

where $G = \{g = (g_1, g_2, \cdots, g_n) : g_1 + g_2 + \cdots g_n = i, g_v$ is zero or a positive integer, all $1, 2, \cdots, n\}$. Now define the set W as $W = \{w = (w_1, \cdots, w_f) : 1 \leqq f \leqq i-1,\ i \geqq w_1 \geqq w_2 \geqq \cdots \geqq w_f \geqq 1,\ w_1 \geqq 2,\ w_1 + w_2 + \cdots + w_f = i\}$, and $G(w)$ for each element of the set W as follows: $G(w) = \{g : g \in G, g = (g_{h_1}, g_{h_2}, \cdots, g_{h_n}) = (w, 0, \cdots, 0)$ for some permutation $(1, 2, \cdots, n) \to (h_1, h_2, \cdots, h_n)\}$. Also, define $G_0 = \{g : g_v = 0$ or 1 for all $v = 1, 2, \cdots, n, g_1 + \cdots + g_n = i\}$. Then, we have $G = G_0 \cup (\cup_{(w \in W)} G(w))$. It is easy to verify that $\sharp G(w)$, or the number of elements of the set $G(w)$ for a fixed w, is not greater than $\binom{n}{f} f! = n!/(n-f)!$, where f is the dimension of the w. Let us define $E_{n,w}^{(i)} = \sum_{(g \in G(w))} (i!/(g_1!\, g_2! \cdots g_n!)) a_{n1}^{g_1} a_{n2}^{g_2} \cdots a_{nn}^{g_n}$ for all $w \in W$. If $g \in G(w)$ and $w \in W$, then, by the definition of $G(w)$, we have $g_1! \cdots g_n! = g_{h_1}! \cdots g_{h_f}!$. Hence we have

$$\begin{aligned} E_{n,w}^{(i)} &\leqq \sum_{g \in G(w)} (i!/(g_{h_1}! \cdots g_{h_f}!)) \left(a_{\max}^{(n)}\right)^i \\ &\leqq \sharp G(w)(i!) \left(a_{\max}^{(n)}\right)^i \\ &\leqq (n!/(n-f)!)(i!) \left(a_{\max}^{(n)}\right)^i \\ &\leqq n^f\, i! \left(a_{\max}^{(n)}\right)^i. \end{aligned}$$

$\sharp W$ is not greater than i^i since $w_v \leqq i$ for all $v = 1, 2, \cdots, f$ and $f \leqq i-1$. Therefore by defining $T_n^{(i)} = \sum_{(c \in CH_{n,i})} i!\, a_{nc_1} a_{nc_2} \cdots a_{nc_i}$, we have

$$T_n^{(i)} \leqq (E_n)^i = E^i = T_n^{(i)} + \sum_{w \in W} E_{n,w}^{(i)} \leqq T_n^{(i)} + i^i \, n^{\tilde{f}} \, i! \left(a_{\max}^{(n)}\right)^i,$$

where $\tilde{f} = \max_{(w \in W)} f$. Since $T_n^{(i)} \geqq \binom{n}{i} i! \left(a_{\min}^{(n)}\right)^i$, we have

$$T_n^{(i)} \leqq E^i \leqq T_n^{(i)} \left(1 + \left(i^i \, n^{\tilde{f}} \, i! \left(a_{\max}^{(n)}\right)^i \Big/ \binom{n}{i} i! \left(a_{\min}^{(n)}\right)^i\right)\right). \quad \text{......(16)}$$

By Eq. (16), Assumption (ii) of Lemma 1, and $\tilde{f} \leqq i-1$, we have $\lim_{n\to\infty}(T_n^{(i)}/E^i) = 1$, so that $\lim_{n\to\infty} T_n^{(i)} = E^i$. (Q. E. D.)

Proof of Theorem 3

Theorem 3 can be verified almost directly from Lemma 1. Indeed when we put $a_{n\nu} = m_{n\nu}$ in Lemma 1, Assumptions (i) and (ii) are fulfilled as shown in the following. First, we have $\sum_{\nu=1}^{n} a_{n\nu} = \sum_{\nu=1}^{n} \int_{(\nu-1)/n}^{\nu/n} b(t)dt = \int_0^1 b(t)dt = S$ (const.) so that (i) is satisfied. Second, since $m_{n\nu} \geqq \min_{((\nu-1)/n \leqq t \leqq \nu/n)} b(t)/n$, we have $a_{\min}^{(n)} = \min_{(1 \leqq \nu \leqq n)} m_{n\nu} \geqq \min_{(1 \leqq \nu \leqq n)} (1/n) \min_{((1-\nu)/n \leqq t \leqq \nu/n)} b(t)$, and by Assumption there exists b_1 such that $b_1 > 0$ and $b(t) \geqq b_1$ for all $t, 0 \leqq t \leqq 1$, so that we have $a_{\min}^{(n)} \geqq \min_{(1 \leqq \nu \leqq n)} b_1/n = b_1/n$, and similarly, we have $a_{\max}^{(n)} \leqq b_2/n$. Hence $\lim_{n\to\infty}\left(\left(a_{\max}^{(n)}\right)^i \Big/ \left(n\left(a_{\min}^{(n)}\right)^i\right)\right) \leqq \lim_{n\to\infty} b_2^i/(n b_1^i) = 0$ for all $i = 1, 2, \cdots$. Thus, applying Lemma 1 to this case, we have $\lim_{n\to\infty} R_{ni} = S^i/i!$ for all $i = 1, 2, \cdots$. (Q. E. D.)

6. The k-Stratum Interest for D_n as a Double Sequence and its Convergence to the Exponential Compound Interest

Now let us define *the k-stratum interest for D_n* by $\sum_{i=1}^{k} R_{ni}$ and denote it by B_{nk}. Similarly *the k-stratum interest in the continuous case* is defined to signify $\sum_{i=1}^{k} S^i/i!$. Theorem 3 says that the k-stratum interest for D_n converges to the k-stratum interest in the continuous case as n tends to $+\infty$. Since $\lim_{k\to\infty}\sum_{i=1}^{k} S^i/i! = e^S - 1$, it follows that

$$\lim_{k\to\infty}(\lim_{n\to\infty} B_{nk}) = (\exp S) - 1.^{3)} \quad \text{.................(17)}$$

That the left-hand side of Eq. (17) can be rewritten rigorously as $\lim_{k,n\to\infty} B_{nk}$ is ensured by the following

Theorem 4 If we extend the definition of B_{nk} by $B_{nk} = B_{nn}$ for all $k \geqq n+1$, we have

$$\lim_{k,n\to\infty} B_{nk} = (\exp S) - 1. \quad \text{........................(18)}$$

Lemma 2 B_{nk} converges as $k\to\infty$ uniformly with respect to n.

Proof The proof of Lemma 2 uses the following Theorem A-1 [4, Th. 39, p. 155] : Let a double sequence $\{\beta_{ni}\}$ fulfill the conditions that (i) $|\beta_{ni}| \leqq \beta_i$ for all $n, i = 1, 2, \cdots$, where β_i's are positive constants, and that (ii) $\lim_{k\to\infty}\sum_{i=1}^{k} \beta_i$ converges. Then, $\lim_{k\to\infty}\sum_{i=1}^{k} \beta_{ni}$ converges uniformly with respect to n. We can apply Theorem A-1 by putting $\beta_{ni} = R_{ni}$ and defining $\beta_{ni} = 0$ for all $i \geqq n+1$. By (10) and (16) in the proof of Lemma 1, $R_{ni} \leqq S^i/i!$ for all $n = 1, 2, \cdots$, fulfilling (i), so that we have $\lim_{k\to\infty}\sum_{i=1}^{k} R_{ni} \leqq (\exp S) - 1 < +\infty$. Thus the conditions (i) and (ii) of Theorem A-1 are both fulfilled. Hence, by Theorem A-1, $B_{nk} = \sum_{i=1}^{k} R_{ni}$ converges uniformly as $k\to\infty$. (Q. E. D.)

Proof of Theorem 4

Let us proceed to the proof of Theorem 4. Here, we use the following Theorem A-2 [4, Th. 44, p. 172] : Let a double sequence $\{\gamma_{nk}\}$ fulfill the conditions that (i) γ_{nk} converges to γ_n as $k\to\infty$ uniformly with respect to n, and that (ii) $\lim_{n\to\infty}\gamma_n = \gamma$. Then, $\lim_{k,n\to\infty}\gamma_{nk}$ exists and equals γ. Putting $\gamma_{nk} = B_{nk}$ for $k, n = 1, 2, \cdots$, we can apply Theorem A-2. By the extended definition of B_{nk}, we have $\lim_{k\to\infty} B_{nk} = B_{nn}$, so that the condition (i) is fulfilled. Finally, by Eq. (13), we have $\lim_{n\to\infty} B_{nn} = (\exp S) - 1$, so that the condition(ii) is fulfilled. Hence, by Theorem A-2, we have $\lim_{k,n\to\infty}$

3) It follows from Theorem 1 that $\lim_{n\to\infty}(\lim_{k\to\infty} B_{nk}) = (\exp S) - 1$ with k and n exchanged in Eq. (17).

$B_{nk}=(\exp S)-1$. (Q. E. D.)

Theorem 4 says that the exponential compound interest is the limit of the k-stratum interest for D_n as a double sequence with respect to k and n.[4]

Conclusion

In this paper the concept of the i-th interest stratum for D_n was introduced, which signifies the i-th stage of the whole cumulative interest-generating process. E. g., the 2nd and 3rd interest strata for D_n mean the totals of all the interests on interests and all the interests on interests on interests, respectively, produced through the n periods on the constant unit principal. The k-stratum interest for D_n is simply the total of the 1st, 2nd, ..., and k-th interest strata, signifying the sum of interests covering only up to the k-th stage of the whole interest-generating process.

The main results can be restated in terms of the k-stratum interest for D_n. First, the compound interest for D_n is the n-stratum interest for D_n, and Theorem 1 proved that the limit of the compound interest for D_n is the exponential compound interest. Second, being based on Theorem 2, Theorem 3 substantially proved that the k-stratum interest for D_n converges to the k-stratum interest in the continuous case as n tends to infinity. Therefore we can make the k-stratum interest for D_n approach the exponential compound interest by first taking n to $+\infty$ and then k to $+\infty$. However can't we converge the former, B_{nk}, to the latter, e^S-1, with both k and n kept to be finite and without taking k equal to n? And if we can, in what way? Theorem 4 shows that there is the very simple way in which we can do that. It follows from Theorem 4 that in order for us to converge B_{nk} to e^S-1 we have only to think of such k as less than n and to take k sufficiently large.

4) This means that, for each arbitrarily given $\varepsilon>0$, there exists a positive integer N_ε such that if $k>N_\varepsilon$ and $n>N_\varepsilon$ we always have $|B_{nk}-((\exp S)-1)|<\varepsilon$ [4, p. 171].

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