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Introduction

The names of von Neumann and Morgenstern have been closely associated with the concept "expected utility" in economics. The apparently casual weighted average of cardinal (as against ordinal) levels of utility by probabilities has recently been often used as a theoretical tool in economic analysis.

The Cowles Foundation Monograph "Portfolio Selection" by H. M. Markowitz [9], on the other hand, describes how portfolio theory can be built on the basis of the von Neumann-Morgenstern theorem. The works on risk by, e.g., Friedman and Savage [4], Tobin [16], Arrow [1], [2], Hirshleifer [6], Yaari [18], and Pratt [13] have formed a substantial development in Neoclassical economic theory. And a theoretical starting point of this development is also the theory of expected utility presented in "Theory of Games and Economic Behavior" by von Neumann and Morgenstern [17].

Apart from viewing it as a foundation of the developments just mentioned, the original von Neumann-Morgenstern theorem on utility has by itself the striking theoretical appeal that any set of 'states' or 'commodity (or consumption) bundles' which are completely ordered only in the ordinal sense can be given a cardinal measure of utility of its elements, provided a set of axioms is accepted with respect to the prefenence ordering among sure consumption bundles and multi-stage probability combinations of the sure consumption bundles. According to the theorem, if the axioms are fulfilled, the indifferent 'curves' can be put numerical values, 2 utils (say), 5 utils, 7 utils, etc., clearly indicating whether the marginal utility is increasing or decreasing. This is a remarkable conclusion to anyone who have been taught that indifference 'curves' (e. g.,) are just like contour lines in a map any of which is not known to correspond to 'how high' a level of utility, and that even the difference in utility between an indifference curve A and another indifference curve B is not known to be greater or less than the difference in utility between the indifference curve B and some third indifference curve C.

Even if we are given only a preference ordering on the set of sure prospects, each indifference curve can be put a numerical value in such a way that the 'more' preferred indifference curve is put a greater value than the 'less' preferred. Such a measure, however, is not uniquely determined so as for it to be invariant with respect to convexity.

Given the idea of a probability combination (or a 'mixture') of two sure comsumption bundles (e. g.,) which belong to two different indifference curves A and B resp., and given the economic agent's ability to evaluate such a mixture as equivalent in utility to a sure consumption bundle which belongs to some third indifference curve C, the three indifference curves will become numerically comparable in utility in the sense that the difference in utility between A and C is in such and such a proportion to the difference in utility between C and B. The very values of the probabilities pertaining to the mixture of the first two sure consumption bundles will help give the ratio.

This idea of their theorem is therefore simple, and based on this idea, their theorem contributed to economic theory by postulating only a few fairly irresistible axioms about the agent's behavior of rational choice among sure prospects and multi-stage probability mixtures of sure prospects, as a consequence of which some cardinal measure of utility was shown to necessarily exist which is invariant with respect to convexity (or concavity) if definable at all.

It should be remarked that their set of axioms is postulated so as to ensure the calculation of expected utility in the form

$$f((a_1, a_2, a_3, \ldots, a_n \,;\, \alpha_1 : \alpha_2 : \alpha_3 : \ldots : \alpha_n)) = \sum_{k=1}^{n} \alpha_k f(a_k), \ldots (1)$$

where the the argument of the term on the left-hand side denotes the probability mixture which consists of sure prospects $a_1, a_2, \ldots, a_n$ with probabilities $\alpha_1, \alpha_2, \ldots,$ and α_n, attached to each of them, resp., and $f(\cdot)$ denotes the cardinal utility function. The above formula maintains that the cardinal utility of such a probability mixture equals to the weighted average of the cardinal utilities of all $a_k, k=1, 2, \ldots, n$, by $\alpha_1 : \alpha_2 : \ldots : \alpha_n$ ($\sum_{k=1}^{n} \alpha_k = 1$, $\alpha_k > 0$, $k = 1, 2, \ldots, n$.)

One of the aims of von Neumann and Morgenstern's axiomatic approach seems to lie in answering the question how to *write down* rationality (or regularity) rules *in the most concise manner* on which a set of cardinal utility functions can be constructed *so as to ensure that such a calculation as in (1) is always permitted.*

Such a way of writing down rationality rules in 'the most concise' manner is by no means unique. (See [5], e. g., for an alternative set of axioms.)

The scope, or the extents of the quantifications, of the original axioms does not seem to have attracted much attention. The quantifications of the axioms are not specified explicitly in [17]. In this paper the scope of the axioms is specified, and it is clarified that the extents of the quantifications of the original axioms can be significantly boiled down from the universal set U in [17]. (See Section 2, 1-(1) of this paper.)

Another new aspect of this paper lies in the assumed (both up- and downward) 'extremumless' ordering of the set U in the sense that it has neither maximal nor minimal elements. It will be shown that, interestingly, in that case, the original construction is verified entirely without a change in the axioms. The case with maximums and/or minimums is also treated.

Thirdly, the new proof in this paper of the von Neumann-Morgenstern theorem is based on the distinction between sure prospects and lotteries, and this will help clarify how the theorem accommodates the case in which some or all sure prospects are 'discrete' with respect to the preference ordering, as exemplified by the following. (In this example, there happens to be assumed to exist the extremal elements.)

An Example

Let two real numbers x, y, and z, $x<z<y$, be *income levels* in a certain unit of money, and let us consider an economic agent who evaluates utility of income levels x and y to be equal to real numbers u_x and u_y, resp., in some unit of utility.

It is assumed that $u_x>u_y$, so that he evaluates the income y ($>x$) better than the income x. He is assumed to be subject to the von Neumann-Morgenstern's axioms about rational behavior in the probabilistic circumstances. (cf. [17])

Now, assume, further, that he is not sure as to how better off

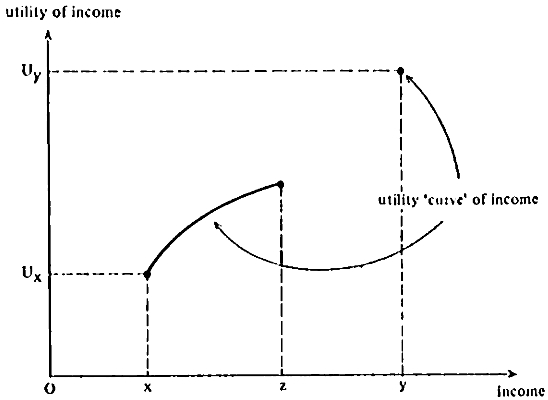

Fig. 1

he would become as his income rised from the level z toward y, because, for example, he has not experienced life with income levels between z and y, etc. Assume, also, that, nonetheless, he knows by how much the life with income y is better than that with income x or z.

He is assumed to know in what pace he becomes better off as his income rises from x to z, not in the sense of some average pace of getting-better-off, but in the sense of exactly how the pace of getting-better-off itself changes as the level of income changes (between x and z.)

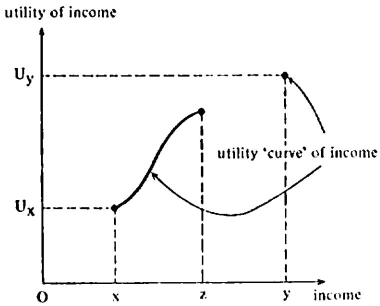

Fig. 2

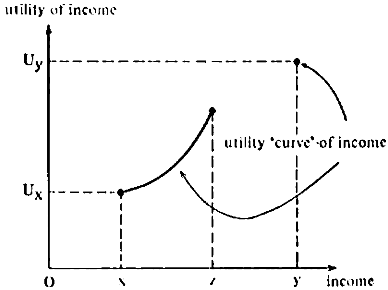

Fig. 3

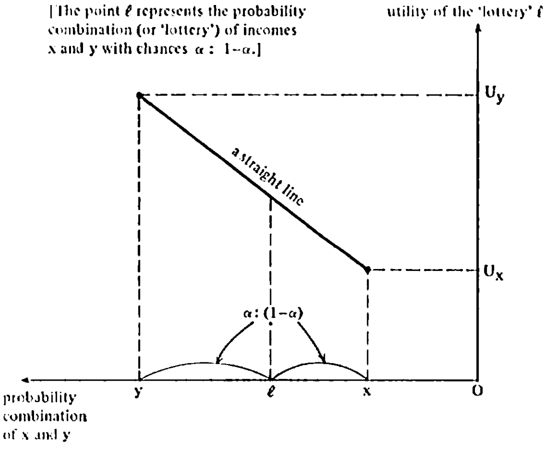

Fig. 4 *the Utility Measure*

Under these assumptions, his utility function of income may be determined like depicted in Fig. 1, as well as possibly like, e.g., in Figs. 2 or 3, though, in theory, only one curve is unambiguously and uniquely determined anyway.

The essential points are that the given utility levels u_x and u_y first give us a utility measure of him as depicted by the straight line in Fig. 4, and that, logically then, it will be determined to what probability combination (i.e., to what l between x and y in Fig. 4) each of income levels between x and z corresponds (or is

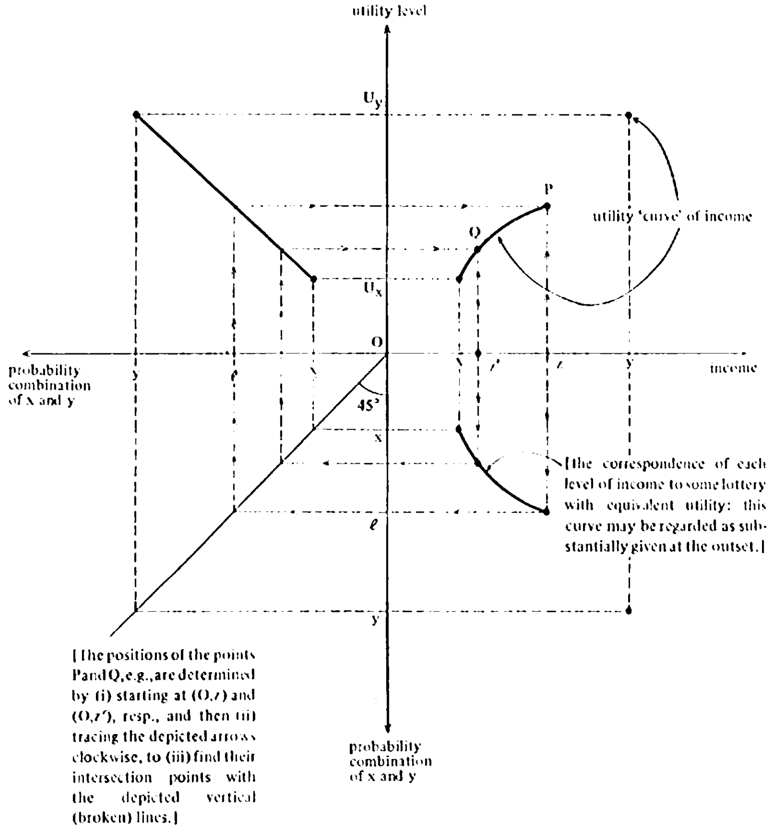

Fig. 5 *How the Utility Curve is Obtained*

equivalent in utility), by the rational behavior of the particular economic agent. (See Fig. 5.)

But why can the utility function be undefined at income levels between z and y? Is such a 'discontinuous' shape of the utility 'curve' consistent with the von Neumann-Morgenstern rationality axioms? Can such a utility curve as in Fig. 1 really be determined uniquely?—According to the analysis in this paper, the answers to these questions except the first are both 'yes.'

Furthermore, in order for it to be possible that such a 'discontinuous' utility curve of income exist and be uniquely determined, the apparent scope of von Neumann-Morgenstern's axioms seems to contain redundancy. This is one of what the present paper clarified. It is also indicated to what extent the scope can be narrowed. Thus, the analysis in this paper may be applied to the case in which there is some probability combination of sure prospects which does not have any counterparting sure prospect.

Section I. Notation and Assumptions

1. Sure Prospects[1] and the Preference Ordering

Let S be an arbitrary set, finite or infinite. Let us call elements of the set S *sure prospects*, and denote them by x, y, z, a, b, c, etc. Let $x \succ y$ denote that *x is preferred to y*, and $x \mathrm{I} y$ that *x is indifferent* or equivalent *to y*. It is assumed that

(1) If $x, y \in S$, then one and only one of the following relations holds: $x \succ y$, $y \succ x$, or $x \mathrm{I} y$.

(2) If x, y and $z \in S$, then $x \mathrm{I} y$ and $y \mathrm{I} z$ imply $x \mathrm{I} z$.

(3) If $x \in S$, then $x \mathrm{I} x$.

(4) If x and $y \in S$, then $x \mathrm{I} y$ implies $y \mathrm{I} x$.

(5) If x, y and $z \in S$, then $x \succ y$ and $y \succ z$ imply $x \succ z$.

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1) The concept of 'sure prospects' is mentioned in Herstein and Milnor [5] and discussed in Marschak [10].

(1) to (5) imply the following (6) and (7):

(6) If x, y and $z \in S$, then $x I y$ and $y \succ z$ imply $x \succ z$.

(7) If x, y and $z \in S$, then $x \succ y$ and $y I z$ imply $x \succ z$.

2. General Prospects

Let T denote the set such that (i) $S \subset T$, and (ii) for any α, $1 > \alpha > 0$, and for any x, $y \in T$, the combination (x, y, α) is an element of T. (x, y, α) is interpreted as the probability combination of x and y with probabilities α and $1-\alpha$, resp. Let L denote the set of all elements (a, b, α), $1 > \alpha > 0$, a, $b \in S$. Any element of the set L is called a *primary lottery*. Let N denote the set of all elements (x, y, α), $1 > \alpha > 0$, x, $y \in S \cup L$. We define that an element of T is a *two-stage lottery* if and only if it belongs to $N-L$. Even such a lottery as $(a, (a, a, \alpha), \beta)$, where $a \in S$, is regarded as a two-stage lottery. Three-stage, or, in general, multi-stage lotteries are defined accordingly. The set T is called as the set of *general prospects*. Any element of T, or any general prospect, x, belongs to one of the three cases: $x \in S$, $x = (y, z, \alpha)$ for some $1 > \alpha > 0$, y, $z \in S$, or x is a *multi-stage lottery*. If x is a multi-stage lottery, then the number of its stages is finite, though there is not any upper bound to the set of numbers of stages.

S, and hence T, *may not* have either maximal or minimal elements.

3. Extensions of the Preference Ordering to General Prospects: the Axioms

The economic agent is supposed to be situated in a probabilistic environment: the comparison in utility he is supposed to make may not only be between sure prospects, but also between a sure prospect and a primary or multi-stage lottery, and between two primary or multi-stage lotteries. Therefore the environment in which he is situated is supposed to include all multi-stage lotteries, i. e., such that

$((x, y, \alpha), (z, t, \beta), \gamma)$, and

$((((x, y, \alpha), (z, t, \beta), \gamma), (u, v, \delta), \varepsilon), (w, s, \psi), \varphi)$,

etc., where x, y, z, etc. belong to S, and α, β, γ, etc. belong to the real open interval (0, 1).

He is supposed to obey the following rules, or 'axioms' on which his whole preference system is entirely characterized, though, it is important to point out, preference systems may differ from one economic agent to another, owing to the freedom of choice the set of axioms permits.

$L \equiv (z : z=(x, y, \alpha),\ x, y \in S,\ 1>\alpha>0)$.

$N \equiv (z : z=(x, y, \alpha),\ x, y \in S \cup L,\ 1>\alpha>0)$.

(a 1) $\forall x,\ y \in S \cup N$,

If $x \succ y$, then $(x, y, \alpha) \succ y$, $\forall \alpha$, $1>\alpha>0$.[1]

(a 2) $\forall x,\ y \in S \cup N$,

If $x \succ y$, then $x \succ (x, y, \alpha)$, $\forall \alpha$, $1>\alpha>0$.

(a 3) $\forall x,\ y,\ z \in S \cup L$,

If $x \succ z \succ y$, then $(x, y, \alpha) \succ z$, $\exists \alpha$, $1>\alpha>0$.[2]

(a 4) $\forall x,\ y,\ z \in S \cup L$,

If $x \succ z \succ y$, then $z \succ (x, y, \alpha)$, $\exists \alpha$, $1>\alpha>0$.

(a 5) $\forall x,\ y,\ t,\ z \in T$,

If $x \mathrm{I} y$ and $t \mathrm{I} z$, then $(x, t, \alpha) \mathrm{I} (y, z, \alpha)$, $\forall \alpha$, $1>\alpha>0$.

(a 6) $\forall x,\ y \in S \cup L$, $\forall \alpha$, $1>\alpha>0$,

$(x, y, \alpha) \mathrm{I} (y, x, 1-\alpha)$.

(a 7) $\forall x,\ y \in S \cup L$, $\forall \alpha$, $\forall \beta$, $1>\alpha>0$, $1>\beta>0$,

$((x, y, \alpha), y, \beta) \mathrm{I} (x, y, \alpha\beta)$.

The axioms (a 1) to (a 4) and (a 6, 7) are in the same forms as those in von Neumann and Morgenstern's original literature except that the scope in which they are required to hold has been

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1) The symbol '$\forall$' means 'for any$\sim$' or, in other words, 'for all$\sim$.' E. g., '$\forall x,\ y \in T$,' means 'for any (all) x and y such that $x, y \in T$,.......'

2) The symbol '$\exists$' means 'there exist(s) some.......' E. g., '$\exists \alpha$, $1>\alpha>0$' means 'there is some α such that $1>\alpha>0$.'

taken to be narrower than the whole set T.

The axiom (a 5) is essentilly what is called Samuelson's strong independence axiom (See, e. g., [14]). This is the only axiom which must be postulated for the whole set T. In this sense, this axiom may be regarded as, so to speak, a 'pre-axiom' as distinct from the other, which may explain why von Neumann and Morgenstern [17] do not write it explicitly as an axiom along with the other. (As to interpretations of the axioms except (a 5), see the literature [17], and as to (a 5) see Malinvaud [8].)

In addition, of course, we have to presume the following:

(b 1) If x, $y \in T$, then one and only one of the following relations holds: $x \succ y$, $y \succ x$, or $x \mathrm{I} y$.

(b 2) If x, y and $z \in T$, then $x \mathrm{I} y$ and $y \mathrm{I} z$ imply $x \mathrm{I} z$.

(b 3) If $x \in T$, then $x \mathrm{I} x$.

(b 4) If x and $y \in T$, then $x \mathrm{I} y$ implies $y \mathrm{I} x$.

(b 5) If x, y and $z \in T$, then $x \succ y$ and $y \succ z$ imply $x \succ z$.

(b 1) to (b 5) imply:

(b 6) If x, y and $z \in T$, then $x \mathrm{I} y$ and $y \succ z$ imply $x \succ z$.

(b 7) If x, y and $z \in T$, then $x \succ y$ and $y \mathrm{I} z$ imply $x \succ z$.

For notational convenience we write $x \succsim y$ or $y \precsim x$ if and only if $x \succ y$ or $x \mathrm{I} y$. All these presumptions (b 1) to (b 5) form no new restrictions or qualifications as compared with the original framework of [17].)

Section II. Logical Motivations to the Analysis

1. Some Relevant Conceptual Distinctions

In order to clarify the logical position of the present investigation as compared with that in [17], two distinctions concerning the set T will have to be made first: namely, the distinction between the set of sure prospects and the set of lotteries and the distinction between the set of equivalence classes of T and the set T itself.

(1) Sure Prospects and Lotteries

The set U in [17][1] exactly corresponds to the concept of the set T as defined above. There, the set U consists of all sure prospects and all lotteries. And T consists of both sure prospects and lotteries. For sure prospects are nothing but elements of the set S, and lotteries are nothing but elements of $T-S$. Thus, $U=T$. The distinction between the set S and the set $T-S$ lies in that S is given first and $T-S$ is, so to speak, generated on the basis of elements of S. E.g., if a and b belong to S, then a and b do not belong to $T-S$, but lotteries (a, b, α), (b, a, α), and even (a, a, α) or (b, b, α), belong to $T-S$, and never belong to S, even though as it will turn out later $(a, a, \alpha)\mathrm{I}a$ and $(b, b, \alpha)\mathrm{I}b$. Here we have to distinguish the relation 'I' and the equality '='. Even if any element of $T-S$ has the relation 'I' (equivalence in utility) to some element of S, it must be regarded as a different element of the whole T from the latter. This is also the case, of course, for any multi-stage lotteries which are equivalent to some elements of S. So, all lotteries including multi-stage lotteries constitute the set $T-S$, and any element of $T-S$ is a (possibly multi-stage) lottery built on the basis of elements of S.

(2) The Set T and the Set of its Equivalence Classes T/I

The axioms, especially (a 5, 6, 7), so to speak, 'put together' elements equivalent to each other to one subset of T. E.g., the axiom (a 6) puts together two elements of N, (x, y, α) and $(y, x, 1-\alpha)$ to a subset of T. If some element of T does not have any other element of T which is equivalent to it, then a subset of T can still be formed as consists of the single element of T. Thus, all elements of T are classified out to such subsets each of which

1) As for the definition of U, it only runs '[w]e consider a system U of entities u, v, w, ... In U, a ***relation*** is given, $u>v$, and for any number α, $(0<\alpha<1)$, an ***operation*** $\alpha u+(1-\alpha)v=w$.' (italic in the original.)(P. 26, [17])

consists of mutually equivalent elements of T and any of such subsets does not have an element which is equivalent to an element which belongs to another of such subsets. Such subsets of T are called "equivalence classes" of T, and the set of equivalence classes of T is denoted by T/I, and called as 'the quotient set of T with respect to I.'

The distinction between the set T and its quotient set T/I is pointed out by Malinvaud [8] in relation to the Samuelson strong independence axiom (a 5).

On the basis of the above two distinctions, the following distinction between two cases should be considered.

(3) The Equivalence Classes of S and of T

Since the relation 'I' is defined on S as well as T, there are also equivelence classes of the set S with respect to I, and the set of such equivalence classes may be written as S/I. We must distinguish between this S/I and another set T/I. Each and every element of S/I which is an equivalence class of S, always corresponds to some element of T/I, which contains an element of T which is equivalent to elements of the element of S/I, since $S \subset T$. However, the converse is not necessarily true. Namely, there can be cases in which some element of T/I which is an equivalence class of T does not have any element of S/I which contains some element of S which is equivalent to its elements.

Consider the following example. $S=$(income of a yen, income of b yen, and income of c yen), $a>b>c>0$, so that S consists of only three elements. The number of sure prospects equals to three. Assume that the economic agent has such preference as $a \succ b$, $b \succ c$, and $a \succ c$. He is assumed to be subject to all the axioms (a 1) to (a 7) along with the presumptions (b 1) to (b 5). Then, the lottery $(a, b, 1/2)$, for example, corresponds to no element of S, i.e., a, b and c, which is equivalent in utility to $(a, b, 1/2)$. This is an example in which some element of T/I does not correspond to any element of S/I.

2. Comparison of the Scopes of the Axioms

If any element of T/I corresponds to an element of S/I, then, in virtue of the Samuelson axiom (a 5), the other axioms than (a 5), will need to be fulfilled only by elements of S, or, in the Malinvaud context [8], by elements of S/I. But otherwise, then even in the presence of (a 5) the other axioms will have to be fulfilled by not only elements of S, or S/I, but also by, at least some of, elements of $T-S$, or $T/I-S/I$. According to the result of the analysis in this paper, the other axioms than the strong independence axiom are sufficient for the same von Neumann-Morgenstern conclusion to be proved, even if they are, as postulates, not required to be fulfilled strictly by all elements of T/I, but if only they are postulated to be fulfilled by all elements of a *proper* subset of $T/I-S/I$, or, specifically, only at most two-stage lotteries, along with all elements of S, or S/I.

Thus, our proof based on the less restrictive axioms implies that the *truely necessary* axiomatic requirements of their original axioms, especially of the forms in (a 6) and (a 7) among the other, are much weaker than the original text might indicate.

As explained in (3) of Subsection 1, the following two cases should be distinguished: (i) the case where any element of T/I 'corresponds' (in the sense explained in (3)) to some element of S/I, and (ii) the case where some element of T/I does not correspond to any element of S/I. In Case (i), the original axioms (with the independence axiom explicit) in [17] and their proof may be interpreted as dealing with elements of S/I (or, equivalently, elements of *T/I represented by those of S/I*). So the situation is clear in this case. But, in Case (ii), some confusion ought to arise from their preference of using the symbol ' = ' in place of ' I ', and from their unspecified extents of the quanifications of the axioms.

For, in this case, the scope of their axioms may be interpreted as comprising all elements of T/I.

3. The Key Formula as a Motivation to the Present Analysis

The formula which is a fundamental device for this paper is as follows:

$$((x, y, \alpha), (x, y, \beta), \gamma)I(x, y, \beta+\gamma(\alpha-\beta))$$

which will be proved in Theorem 5 in its preliminary form and in Theorem 10 in general.

This formula can be derived from our set of axioms before arriving at the von Neumann-Morgenstern conclusion. Therefore this formula is thus derivable also from the original axioms in [17][1)]. It might be interesting that this formula is implied by them so to speak as a 'quasi-axiom' concerning the combining operation, since the formula is derivable without the two axioms (a 4, 5) which are essentially required only in order to prove the continuity theorem (Theorem 1, below).

Section III. Basic Theorems

Theorem 1. $\forall x, y, z \in S$, if $x \succ z \succ y$, then $\exists \alpha$, $1 > \alpha > 0$, $zI(x, y, \alpha)$.

Proof. By (a 3), $\exists \alpha_1$, $1 > \alpha_1 > 0$, $(x, y, \alpha_1) \succ z \succ y$.......(1)

By (a 3), $\exists \alpha_2'$, $1 > \alpha_2' > 0$, $((x, y, \alpha_1), y, \alpha_2') \succ z \succ y$. By (a 7), $(x, y, \alpha_1 \alpha_2') \succ z \succ y$.......(2)

1) They say '[d]o not our postulates introduce, in some oblique way, the hypotheses which bring in the mathematical expectation?' (l. 9, P. 28, [17]) and '[t]he deductions which follow....are rather lengthy.... [T]he ideas that underly the deductions are quite simple, but unfortunately the technical execution had to be somewhat voluminous in order to be complete. Possibly a shorter exposition might be found later' (P. 617-8, [17]) just before their proof. Though the following proof cannot be claimed to be shorter as a whole than their original, it aims to improve upon it by using the above formula in naturally splitting the proof into three stages.

Putting $\alpha_2=\alpha_1\alpha_2'$, we have $1>\alpha_2>0$, so that, by Eq. (2) and by the same reasoning as in getting Eq. (2) from Eq. (1), we have $(x, y, \alpha_3)\succ z\succ y$, $\alpha_3=\alpha_2\alpha_3'$. Repeating this step, we have $\alpha_1>\alpha_2>\alpha_3>\ldots\ldots>\alpha_m>\ldots\ldots$(3)

The series (α_m) is a monotonely decreasing series with a lower bound 0, so that it has its limit, say $\underline{\alpha}$, $1>\underline{\alpha}\geqq 0$.

By (a 4), $\exists\beta_1$, $1>\beta_1>0$, $x\succ z\succ(x, y, \beta_1)$.......(1′)

By (a 4), $\exists\beta_2'$, $1>\beta_2'>0$, $z\succ(x, (x, y, \beta_1), \beta_2')$. By (a 5, 6, 7), (the right-hand side)$\mathrm{I}(x, y, \beta_1+\beta_2'-\beta_1\beta_2')$. Hence, $x\succ z\succ(x, y, \beta_2)$, $\beta_2\equiv\beta_1+\beta_2'-\beta_1\beta_2'$.......(2′)

Repeating the same reasoning as in getting Eq. (2′) from (1′), we get a monotonely increasing series (β_m) with an upper bound 1. Hence, the series has the limit $\bar{\beta}$, $1\geqq\bar{\beta}>0$.

Suppose $\underline{\alpha}=0$. Then $(x, y, \alpha)\succ z\succ y$ for some α such that $\bar{\beta}>\alpha\geqq 0$. By the property of $\bar{\beta}$, $z\succ(x, y, \beta)$ for some β such that $\bar{\beta}\geqq\beta>\alpha$. Hence, by (a 1), $(x, y, \alpha)\succ(x, y, \beta)\succ y$.......(1″)

By (a 7), $(x, y, \alpha)\mathrm{I}((x, y, \beta), y, \alpha/\beta)$.......(2″)

Since $(x, y, \beta)\succ y$, by (a 2), $(x, y, \beta)\succ((x, y, \beta), y, \alpha/\beta)$......(3″)

By (1″), (2″) and (3″), we have $(x, y, \beta)\succ(x, y, \beta)$, a contradiction. Hence, $\underline{\alpha}>0$.

Suppose $\bar{\beta}=1$. Then $\bar{\beta}>\underline{\alpha}$. By the property of $\bar{\beta}$, there exists β, $\bar{\beta}>\beta>\underline{\alpha}$, $x\succ z\succ(x, y, \beta)$. By the property of $\underline{\alpha}$, there exists α, $\beta>\alpha\geqq\underline{\alpha}$, $(x, y, \alpha)\succ z$. Hence, we have $x\succ(x, y, \alpha)\succ(x, y, \beta)$, $\beta>\alpha$ ......(i). But by $\beta>\alpha$ and (a 5, 6, 7), we have $(x, y, \beta)\mathrm{I}(x, (x, y, \alpha), (\beta-\alpha)/(1-\alpha))$.......(ii). Since $x\succ(x, y, \alpha)$, we have, by (a 1), $(x, (x, y, \alpha), (\beta-\alpha)/(1-\alpha))\succ(x, y, \alpha)$.......(iii). By (i), (ii), (iii), we have $(x, y, \alpha)\succ(x, y, \alpha)$, a contradiction. Hence $\bar{\beta}<1$.

There may exist more than one α_1 which satisfy Eq. (1). Let A_1 denote the set of all such α_1. For each $\alpha_1\in A_1$, there exists the series (α_m), and its limit $\underline{\alpha}$. Let us put this as $\underline{\alpha}(\alpha_1)$. The set $(\underline{\alpha}(\alpha_1), \alpha_1\in A_1)$ is bounded from below, so that it has its infimum $\underline{\underline{\alpha}}$ (Weierstrass). $1>\underline{\underline{\alpha}}\geqq 0$. For any positive number ε, there exists some $\alpha_1\in A_1$ such that $\underline{\underline{\alpha}}+\varepsilon>\underline{\alpha}(\alpha_1)>0$. By the property of $\underline{\alpha}(\alpha_1)$, there exists some m such that $\underline{a}+\varepsilon>\alpha_m\geqq\underline{a}(\alpha_1)$, where α_m is a term of the series (α_m) corresponding to the α_1.

Now, suppose $\underline{a}=0$. Then, we can take $\varepsilon=\bar{\beta}$ in the above argument, and there exists α such that $(x, y, \alpha)\succ z\succ y$, and $\bar{\beta}>\alpha>0$. By the property of $\bar{\beta}$ there exists some β such that $\bar{\beta}\geqq\beta>\alpha$ and $z\succ(x, y, \beta)$. By the same argument about Eqs. (1″) to (3″) above, a contradiction is obtained. Hence, $\underline{a}>0$. By the definition of an infimum, $\underline{a}$ is uniquely determined from the triplet (x, y, z). Since $1>\underline{a}>0$, $(x, y, \underline{a})$ is already included in the set L.

If $(x, y, \underline{a})\succ z$, then $(x, y, \underline{a})\succ z\succ y$, so that $\underline{a}\in A_1$, we get the decreasing series (α_m), $\alpha_1=\underline{a}$, and have $\alpha_2<\underline{a}$, $\alpha_2\in A_1$, a contradiction. If $z\succ(x, y, \underline{a})$, then $x\succ z\succ(x, y, \underline{a})$, so that by the same argument about Eqs. (1′) and (2′), we get the series (β_m') with $\beta_1'=\underline{a}$ and its limit $\bar{\beta}'$. Then we have $\bar{\beta}'>\underline{a}$, and, by the property of $\underline{a}$, there exists some α such that $\bar{\beta}'>\alpha\geqq\underline{a}>0$, $(x, y, \alpha)\succ z$. By the property of $\bar{\beta}'$, there exists some β such that $1>\bar{\beta}'\geqq\beta>\alpha$, $z\succ(x, y, \beta)$. Hence, $(x, y, \alpha)\succ(x, y, \beta)\succ y$. By (a 7), $(x, y, \alpha)\mathrm{I}((x, y, \beta), y, \alpha/\beta)$. By (a 2), $(x, y, \alpha)\succ((x, y, \beta), y, \alpha/\beta)$. Hence, $(x, y, \alpha)\succ(x, y, \alpha)$, a contradiction. Therefore, we must have $(x, y, \underline{a})\mathrm{I}z$.

Theorem 2. $\forall x\in S\cup L$, $\forall\alpha$, $1>\alpha>0$, it holds that $(x, x, \alpha)\mathrm{I}x$.

Proof. (along the line of von Neumann-Morgenstern's proof of their proposition $(A:T)$ in [17].) Suppose $(x, x, \alpha)\succ x$, where $x\in S\cup L$. Then, noting that $(x, x, \alpha)\in S\cup N$, we can apply (a 1, 2), and get $(x, x, \alpha)\succ((x, x, \alpha), x, \beta)\succ x$ for any β, $1>\beta>0$. Since $x\in S\cup L$, we can apply (a 7) to the second term, and get $(x, x, \alpha)\succ(x, x, \alpha\beta)\succ x$. Hence, $(x, x, \alpha)\succ(x, x, \gamma)\succ x$ for any γ, $\alpha>\gamma>0$.......(1). Since $x\in S\cup L$, we can apply (a 6), and get $(x, x, \alpha)\mathrm{I}(x, x, 1-\alpha)$ and $(x, x, \gamma)\mathrm{I}(x, x, 1-\gamma)$. By (1), we have $(x, x, 1-\alpha)\succ(x, x, 1-\gamma)\succ x$ for any $\gamma, \alpha>\gamma>0$. Therefore, if $(x, x, \alpha)\succ x$, then we have $(x, x, \alpha)\mathrm{I}(x, x, \alpha')\succ(x, x, \gamma')\succ x$ for any γ', $\gamma'>\alpha'$,......(2), where $\alpha'=1-\alpha$. But by $(x, x, \alpha')\succ x$, we can substitute α' *for* α in (2), and get $(x, x, \alpha')\mathrm{I}(x, x, 1-\alpha')\succ(x, x, \gamma^\circ)\succ x$ for any γ°, $\gamma^\circ>1-\alpha'$. Since $\alpha=1-\alpha'$, we have $(x, x, \alpha)\succ(x, x, \gamma^\circ)\succ x$ for any γ°, $\gamma^\circ>\alpha$.......(3).

If $(x, x, \alpha) \prec x$ for some α, $1 > \alpha > 0$, then, by (a 1, 2), $x \succ (x, (x, x, \alpha), 1-\beta) \succ (x, x, \alpha)$ for any β, $1 > \beta > 0$. Hence, by (a 6, 7), $x \succ (x, x, \alpha\beta) \succ (x, x, \alpha)$ for any β, $1 > \beta > 0$, so that we have $(x, x, \alpha) \prec (x, x, \gamma) \prec x$, $\forall \gamma$, $1 > \alpha > \gamma > 0$.......(4). By (4) and (a 6), we have $(x, x, 1-\alpha) \prec (x, x, 1-\gamma) \prec x$ for any γ, $1 > \alpha > \gamma > 0$. Therefore, if $(x, x, \alpha) \prec x$, we have, by putting $\gamma' = 1-\gamma$, $(x, x, \alpha) \mathrm{I} (x, x, \alpha') \prec (x, x, \gamma') \prec x$ for any γ', $\alpha' < \gamma'$,......(5), where $\alpha' = 1-\alpha$. But by $(x, x, \alpha') \prec x$, we can substitute α' *for* α in (5), and get $(x, x, \alpha') \mathrm{I} (x, x, 1-\alpha') \prec (x, x, \gamma^\circ) \prec x$ for any γ°, $1-\alpha' < \gamma^\circ$. Since $\alpha = 1-\alpha'$, we have $(x, x, \alpha) \prec (x, x, \gamma^\circ) \prec x$ for any γ°, $\alpha < \gamma^\circ$.......(6).

Now, suppose $(x, x, \alpha_1) \succ x$ for some α_1, $1 > \alpha_1 > 0$. Then, by (1), we have $(x, x, \alpha_1) \succ (x, x, \gamma) \succ x$ for any γ, $\alpha_1 > \gamma$. However, if we fix such α_1 and γ, then, by (3), we also have $(x, x, \gamma) \succ (x, x, \alpha_1)$ as follows: since $(x, x, \gamma) \succ x$, we have (1) and (3) for the case $\alpha = \gamma$, so that we may put $\gamma^\circ = \alpha_1$ in this (3), and have $(x, x, \gamma) \succ (x, x, \alpha_1)$ by this (3). Hence $(x, x, \alpha_1) \succ (x, x, \alpha_1)$, a contradiction. Hence, not $(x, x, \alpha) \succ x$ for any α, $1 > \alpha > 0$.

Suppose, therefore, that $(x, x, \alpha_1) \prec x$. Then, by (4), we have $(x, x, \alpha_1) \prec (x, x, \gamma) \prec x$ for any γ, $\gamma < \alpha_1$. Also, since $(x, x, \gamma) \prec x$, we have (4) and (6) for the case $\alpha = \gamma$, so that we may put $\gamma^\circ = \alpha_1$ in this (6), and have $(x, x, \gamma) \prec (x, x, \alpha_1)$. Hence, $(x, x, \alpha_1) \prec (x, x, \alpha_1)$, a contradiction. Hence, not $(x, x, \alpha) \prec x$ for any α, $1 > \alpha > 0$. It follows that $(x, x, \alpha) \mathrm{I} x$ for all α, $1 > \alpha > 0$.

Lemma 1. $\forall x$, $y \in S$, $x \succ y$, if $\alpha > \alpha'$, then $(x, y, \alpha) \succ (x, y, \alpha')$.

Proof. Suppose $\alpha > \alpha'$, x, $y \in S$. Then, by (a 7), $(x, y, \alpha') \mathrm{I} ((x, y, \alpha), y, \alpha'/\alpha)$. By (a 1), $(x, y, \alpha) \succ y$. By (a 2), $(x, y, \alpha) \succ ((x, y, \alpha), y, \alpha'/\alpha)$. Hence $(x, y, \alpha) \succ (x, y, \alpha')$.

By Lemma 1 and (a 6), it is easy to see that if not $x \mathrm{I} y$ and $\alpha \neq \alpha'$ then not $(x, y, \alpha) \mathrm{I} (x, y, \alpha')$. Therefore, we have

Theorem 3. $\forall x$, y, $z \in S$, if $x \succ z \succ y$, then there exists some α uniquely, such that $1 > \alpha > 0$, $z \mathrm{I} (x, y, \alpha)$.

Proof. Suppose $z \mathrm{I} (x, y, \alpha_i)$, $(i = 1, 2)$. Then $(x, y, \alpha_1) \mathrm{I} (x, y, \alpha_2)$, so that $\alpha_1 = \alpha_2$.

Theorem 4. $\forall x, y \in S$, $x \succ y$, we have $(x, y, \alpha) \succ (x, y, \alpha')$ if and only if $\alpha > \alpha'$.

Proof. Suppose $(x, y, \alpha) \succ (x, y, \alpha')$ and $\alpha' > \alpha$. Then by Lemma 1 $(x, y, \alpha') \succ (x, y, \alpha)$, a contradiction. Suppose, instead, that $\alpha' = \alpha$. Then, $(x, y, \alpha') = (x, y, \alpha) \succ (x, y, \alpha')$, a contradiction.

Theorem 5. $\forall x, y \in S$, $\forall \alpha, \beta, \gamma$, $1 > \alpha > 0$, $1 > \beta > 0$, $1 > \gamma > 0$, it holds that $((x, y, \alpha), (x, y, \beta), \gamma) \mathrm{I} (x, y, \beta + \gamma(\alpha - \beta))$.

Proof. First, suppose $\alpha \neq \beta$. Let $1 > \theta > 0$. Substitute x, $(x, y, \theta) \in S \cup L$ for x, y, resp., and ρ, ϕ for α, β, resp. in (a 7). Then, we have

$$((x, (x, y, \theta), \rho), (x, y, \theta), \phi) \mathrm{I} (x, (x, y, \theta), \rho\phi). \quad \cdots\cdots\cdots (1)$$

By (a 5, 6, 7), (the left-hand side of (1)) $\mathrm{I}((x, y, \theta + \rho - \theta\rho), (x, y, \theta), \phi)$, whereas (the right-hand side of (1)) $\mathrm{I}(x, y, \theta + \rho\phi - \theta\rho\phi)$. Hence, for $\forall \theta, \rho, \phi$, $1 > \theta, \rho, \phi > 0$,

$$((x, y, \theta + \rho - \theta\rho), (x, y, \theta), \phi) \mathrm{I} (x, y, \theta + \rho\phi - \theta\rho\phi) \cdots\cdots\cdots (2)$$

(i) Suppose $1 > \alpha > \beta > 0$. Then we can take θ, ρ, ϕ such that $1 > \theta > 0$, $1 > \rho > 0$, $1 > \phi > 0$, which fulfil

$$\theta + \rho - \theta\rho = \alpha, \ \theta = \beta, \text{ and } \phi = \gamma, \ \alpha > \beta, \ 1 > \alpha > 0,$$
$$1 > \beta > 0, \ 1 > \gamma > 0. \quad \cdots\cdots\cdots (3)$$

In fact we get θ, ϕ, $1 > \theta$, $\phi > 0$ by the given β, γ. We get, from the $\theta = \beta$ and the given α, $\rho = (\alpha - \beta)/(1 - \beta)$, where $1 > \rho > 0$. And we get $\theta + \rho\phi - \theta\rho\phi = \beta + \gamma(\alpha - \beta)$. Thus, we have

$$((x, y, \alpha), (x, y, \beta), \gamma) \mathrm{I} (x, y, \beta + \gamma(\alpha - \beta)). \quad \cdots\cdots\cdots (4)$$

(ii) Suppose $1 > \beta > \alpha > 0$. Put $\alpha' = \beta$, $\beta' = \alpha$. Then, since $\alpha' > \beta'$, we have, by (4), $((x, y, \alpha'), (x, y, \beta'), \gamma') \mathrm{I} (x, y, \beta' + \gamma'(\alpha' - \beta'))$. By (a 6), we have $((x, y, \alpha), (x, y, \beta), 1 - \gamma') \mathrm{I} (x, y, \alpha + \gamma'(\beta - \alpha))$. Since γ' is arbitrary $(1 > \gamma' > 0)$, we may take $1 - \gamma' = \gamma$ for the given γ. And we have $((x, y, \alpha), (x, y, \beta), \gamma) \mathrm{I} (x, y, \beta - \gamma(\beta - \alpha))$. Hence the formula holds if $\alpha \neq \beta$.

Suppose $\alpha = \beta$. Then, by (b 3), $(x, y, \alpha) \mathrm{I} (x, y, \beta)$, so that by Theorem 2, we have $((x, y, \alpha), (x, y, \beta), \gamma) \mathrm{I} (x, y, \alpha) \mathrm{I} (x, y, \beta) \mathrm{I} (x, y, \beta + \gamma(\alpha - \beta))$.

Section IV. Theorems on General Prospects

Theorem 6. $\forall x \in T$, $\exists y \in S \cup L$, $x \mathrm{I} y$.

Proof. Let $x \in T$. Suppose there is not any $y \in S$, $x \mathrm{I} y$. Then, x is an element of L, or else, is an at most finite-stage lottery on the basis of a finite number of elements of S. Let S_x denote the set of those basic elements of x. Then, $S_x \subset S$, and S_x contains at least two elements of S which are not equivalent in utility. (Otherwise, all elements of S_x would be equivalent in utility to, say, $y \in S$, so that, by Theorem 2 and (a 5), the multi-stage lottery x would (so to speak) be reduced recursively down to y in the sense of the simple expression $x \mathrm{I} y$, a contradiction.) Since S_x is a finite set, S_x contains both its maximal and minimal elements, denoted m^1 and m_2, resp. If $z \in S_x$, then $z \in S$ and $m^1 \succsim z \succsim m_2$. If $m^1 \succ z \succ m_2$, then, by Theorem 3, we have $z \mathrm{I}(m^1, m_2, \alpha_z)$. Hence, if $z \in S_x$, then one, and only one of $z \mathrm{I} m^1$, $z \mathrm{I} m_2$, or $z \mathrm{I}(m^1, m_2, \alpha_z)$ must hold.

Suppose $x \in S \cup L$. Then, by $x \mathrm{I} x$, there is not anything to prove. So, suppose $x \notin S \cup L$. Then, the raw expression of x as a multi-(finite-) stage lottery includes such forms as $x = (\ldots\ldots, (z_1, z_2, \alpha), \ldots\ldots)$, or $(\ldots\ldots z_1 \ldots\ldots)$, $z_i \in S_x$. Clearly, if $m^1 \succ z_i \succ m_2$, $i = 1, 2$, then $z_i \mathrm{I}(m^1, m_2, \alpha_{z_i})$, $i = 1, 2$, so that, by Theorem 5, we have $(z_i, z_j, \alpha) \mathrm{I}(m^1, m_2, \beta)$, where $\beta = \alpha_{z_j} + \alpha(\alpha_{z_i} - \alpha_{z_j})$. If z_i appears in the raw expression of x as $x = (\ldots\ldots z_i \ldots\ldots)$, we remark that such a z_i can be substituted in the expression by (m^1, m_2, α_{z_i}) owing to (a 5), provided neither $z_i \mathrm{I} m^1$ nor $z_i \mathrm{I} m_2$.

If only one of $m^1 \succ z_i \succ m_2$ and $m^1 \succ z_j \succ m_2$ holds in the form (z_i, z_j, α), then the reduction of (z_i, z_j, α) to the form (m^1, m_2, β) can be ensured by (a 5, 6, 7). If neither $m^1 \succ z_i \succ m_2$ nor $m^1 \succ z_j \succ m_2$, then, clearly $(z_i, z_j, \alpha) \mathrm{I}(m^1, m_2, \beta)$, or $\mathrm{I} m^1$, or $\mathrm{I} m_2$.

Now x may contain in its raw expression such forms as indicated by $x = (\ldots\ldots((z_1, z_2, \alpha), (z_3, z_4, \beta), \gamma)\ldots\ldots)\ldots\ldots(1)$, or $x = (\ldots\ldots$

$((z_1, z_2, \alpha), z_3, \beta)$......), or (......$(z_1, (z_2, z_3, \alpha), \beta)$......).......(2). In any of these cases, by (a 5, 6, 7) and Theorems 2 and 5, the explicitly written part in (1) or (2) will be reduced to the form (m^1, m_2, γ) or m^1 or m_2. Thus, x can be rewritten as xI(......(m^1, m_2, γ)......), or xI(......m^1......), or I(......m_2......) in any of these cases. Similarly, by Theorem 5, the multi-stage lottery x rewritten on the basis of m^1 and m_2 can be recursively reduced its number of stages, finally to arrive at the relation $x\mathrm{I}(m^1, m_2, \theta)$. Since (m^1, m_2, θ), m^1, $m_2 \in S \cup L$, the proof is complete.

Lemma 2. $\forall x \in T$, it holds that $(x, x, \alpha)\mathrm{I}x$ for $\forall \alpha$, $1 > \alpha > 0$.

Proof. By Theorem 6, we have some $y \in S \cup L$ such that $x\mathrm{I}y$. By (a 5), we have $(x, x, \alpha)\mathrm{I}(y, y, \alpha)$, and by Theorem 2, we have $(y, y, \alpha)\mathrm{I}y$. Hence $(x, x, \alpha)\mathrm{I}y\mathrm{I}x$.

Theorem 7. $\forall x, y, z \in T$, $x \succ z \succ y$, there exists α uniquely, $1 > \alpha > 0$, $z\mathrm{I}(x, y, \alpha)$.

Proof. By Theorem 6, $x\mathrm{I}(x_1, x_2, \alpha)$ or $\mathrm{I}x_1$, and $y\mathrm{I}(y_1, y_2, \beta)$ or $\mathrm{I}y_1$, and $z\mathrm{I}(z_1, z_2, \gamma)$ or $\mathrm{I}z_1$, where $x_i, y_i, z_i \in S$, $i = 1, 2$. Let $Q_{(x, y, z)}$, or $Q_{(\ )}$ for brevity, denote the set of x_i, y_i, and z_i which appear in the three expressions. $Q_{(\ )}$ consists of at most six elements, so that it has its maximal and minimal elements, denoted by m^1, m_2. Clearly $m^1 \succ m_2$. (Otherwise, $x\mathrm{I}y\mathrm{I}z$, a contradiction.) By Theorem 1, any element of $Q_{(\ )}$ is equivalent in utility to some primary lottery (m^1, m_2, α) or else to m^1 or m_2. Hence, any of x_i, y_i, and z_i can be reduced to its equivalent, which are m^1, m_2, or of the form (m^1, m_2, α). Clearly, by Theorem 3, the α's are uniquely determined. By (a 5, 6, 7) and Theorem 5, x, y, and z can be reduced to their equivalent of the similar forms. Suppose x and y are reduced to their equivalent, (m^1, m_2, α_x) and (m^1, m_2, α_y), resp. Then, by $x \succ y$ and Theorem 4, we have $\alpha_x > \alpha_y$. Since $z\mathrm{I}(m^1, m_2, \alpha_z)$, we have, by Theorem 4, $\alpha_x > \alpha_z > \alpha_y$. If we take $\gamma = (\alpha_z - \alpha_y)/(\alpha_x - \alpha_y)$, then, by Theorem 5, we have $z\mathrm{I}(m^1, m_2, \alpha_z)\mathrm{I}((m^1, m_2, \alpha_x), (m^1, m_2, \alpha_y), \gamma)\mathrm{I}(x, y, \gamma)$. Suppose $x\mathrm{I}(m^1, m_2, \alpha_x)$ and $y\mathrm{I}m_2$. Then, since $z\mathrm{I}(m^1, m_2, \alpha_z)$, we have, by Theorem 4, $\alpha_z < \alpha_x$, so that, by (a 7), we have $zI(m^1, m_2, \alpha_z)I((m^1, m_2, \alpha_x), m_2, \alpha_z/\alpha_x)I(x, y, \alpha_z/\alpha_x)$. Suppose xIm^1 and $yI(m^1, m_2, \alpha_y)$. Then, by Theorem 4, we have $\alpha_z > \alpha_y$, so that, by (a 5, 6, 7), we have $(x, y, (\alpha_z - \alpha_y)/(1-\alpha_y))I(m^1, (m^1, m_2, \alpha_y), (\alpha_z - \alpha_y)/(1-\alpha_y))I(m^1, m_2, \alpha_z)Iz$. Suppose xIm^1 and yIm_2. Then $zI(m^1, m_2, \alpha_z)I(x, y, \alpha_z)$.

Theorem 8. $\forall x, y \in T$, if $x \succ y$, then $\forall \alpha$, $1 > \alpha > 0$, $x \succ (x, y, \alpha) \succ y$.

Proof. We define the set $Q_{(x, y)}$ just as we defined $Q_{(x, y, z)}$ in the proof of Theorem 7. We denote maximal and minimal elements of $Q_{(x, y)}$ by m^1 and m_2, resp. There can be four cases; (i) xIm^1 and yIm_2, (ii) xIm^1 and not yIm_2, (iii) not xIm^1 but yIm_2, and (iv) neither xIm^1 nor yIm_2. In Case (i), by (a 1, 2), $x \succ (x, y, \alpha) \succ y$. In Case (ii), we have $m^1 \succ y \succ m_2$, so that by Theorem 7 we have $yI(m^1, m_2, \alpha_y)$. By (a 5, 6, 7), we have $(x, y, \alpha)I(m^1, (m^1, m_2, \alpha_y), \alpha)I(m^1, m_2, \alpha_y + \alpha - \alpha_y\alpha)$. Since $\alpha_y + \alpha - \alpha_y\alpha > \alpha_y$, by Theorem 4, we have $(x, y, \alpha) \succ y$. Also by (a 2), we have $xIm^1 \succ (x, y, \alpha)$. The proof in Case (iii) is similar to this case. In Case (iv), $xI(m^1, m_2, \alpha_x)$ and $yI(m^1, m_2, \alpha_y)$, so that by (a 5) and Theorem 5, $(x, y, \alpha)I(m^1, m_2, \alpha_y + \alpha(\alpha_x - \alpha_y))$. By Theorem 4, $\alpha_x > \alpha_y$, so that we have $\alpha_x > \alpha_y + \alpha(\alpha_x - \alpha_y) > \alpha_y$. Hence by Theorem 4, we have $x \succ (x, y, \alpha) \succ y$.

Lemma 3. $\forall x, y \in T$, if $x \succ y$, then $(x, y, \alpha)I(y, x, 1-\alpha)$ for $\forall \alpha$, $1 > \alpha > 0$.

Proof. Since in any case of (i) to (iv) x and y are equivalent to m^1, (m^1, m_2, β), or m_2, which belongs to $S \cup L$. By (a 5, 6), the required conclusion is directly obtained.

Theorem 9. $\forall x, y \in T$, if $x \succ y$, then $(x, y, \alpha_1) \succ (x, y, \alpha_2)$ if and only if $\alpha_1 > \alpha_2$.

Proof. The four cases (i) to (iv) have been defined in the proof of Theorem 8. In Case (i), the conclusion is obtained directly by Theorem 4. Put $z_i = (x, y, \alpha_i)$, $i = 1, 2$. In Case (ii), we have $z_iI(m^1, (m^1, m_2, \alpha_y), \alpha_i)$, so that, by (a 6, 7), we have $z_iI(m^1, m_2, \rho_i)$, where $\rho_i = \alpha_y + \alpha_i - \alpha_y\alpha_i$, $i = 1, 2$. If $z_1 \succ z_2$, then, by Theorem 4, we have $\rho_1 > \rho_2$. Hence $\alpha_1 > \alpha_2$. Conversely, if $\alpha_1 > \alpha_2$, then $\rho_1 > \rho_2$, so that, by Theorem 4, we have $z_1 \succ z_2$. The proof in Case (iii) is similar. In Case (iv), by Theorem 5, we have $z_i\mathrm{I}((m^1, m_2, \alpha_x), (m^1, m_2, \alpha_y), \alpha_i)\mathrm{I}(m^1, m_2, \rho_i)$, where $\rho_i = \alpha_y + \alpha_i(\alpha_x - \alpha_y)$, $i = 1, 2$. By Theorem 4, we have $\alpha_x > \alpha_y$. If $z_1 \succ z_2$, then, by Theorem 4, we have $\rho_1 > \rho_2$, so that we have $\alpha_1 > \alpha_2$. Conversely, if $\alpha_1 > \alpha_2$, then, by $\alpha_x > \alpha_y$, we have $\rho_1 > \rho_2$, so that, by Theorem 4, we have $z_1 > z_2$.

Theorem 10. $\forall x, y \in T$, $\forall \alpha, \beta, \gamma$, $1 > \alpha, \beta, \gamma > 0$, $((x, y, \alpha), (x, y, \beta), \gamma)\mathrm{I}(x, y, \beta + \gamma(\alpha - \beta))$.

Proof. First, suppose $x\mathrm{I}y$. Then, by Lemma 2, we have $(x, y, \alpha)\mathrm{I}x$ for any α. Hence, by (a 5), $((x, y, \alpha), (x, y, \beta), \gamma)\mathrm{I}(x, x, \gamma)$. Again by Lemma 2, $(x, x, \gamma)\mathrm{I}x\mathrm{I}(x, y, \beta + \gamma(\alpha - \beta))$. Hence $((x, y, \alpha), (x, y, \beta), \gamma)\mathrm{I}(x, y, \beta + \gamma(\alpha - \beta))$.

Suppose $x \succ y$. We defined the four cases (i) to (iv) in the proof of Theorem 8. In Case (i), by (a 5) and Theorem 5, we have the required conclusion. In Case (ii), by the argument in the proof of Theorem 8, we have $(x, y, \alpha_i)\mathrm{I}(m^1, m_2, \alpha_y + \alpha_i - \alpha_y\alpha_i)$, $i = 1, 2$. Putting $\rho_i = \alpha_y + \alpha_i - \alpha_y\alpha_i$ $(i = 1, 2)$, we have, by Theorem 5, $((x, y, \alpha_1), (x, y, \alpha_2), \gamma)\mathrm{I}((m^1, m_2, \rho_1), (m^1, m_2, \rho_2), \gamma)\mathrm{I}(m^1, m_2, \rho_2 + \gamma(\rho_1 - \rho_2))$. On the other hand $(x, y, \alpha_2 + \gamma(\alpha_1 - \alpha_2))\mathrm{I}(m^1, (m^1, m_2, \alpha_y), \alpha_2 + \gamma(\alpha_1 - \alpha_2))\mathrm{I}(m^1, m_2, \alpha_y + (\alpha_2 + \gamma(\alpha_1 - \alpha_2)) - \alpha_y(\alpha_2 + \gamma(\alpha_1 - \alpha_2))) \equiv (m^1, m_2, \theta)$. It is easy to see that $\theta = \rho_2 + \gamma(\rho_1 - \rho_2)$. By Theorem 4, putting $\alpha = \alpha_1$ and $\beta = \alpha_2$, we have the required conclusion. The proof in Case (iii) is similar. In Case (iv), by the same reasoning as in the proof of Theorem 8, we have $(x, y, \alpha_i)\mathrm{I}(m^1, m_2, \alpha_y + \alpha_i(\alpha_x - \alpha_y)) \equiv (m^1, m_2, \phi_i)$, $i = 1, 2$. By Theorem 5, we have $((x, y, \alpha_1), (x, y, \alpha_2), \gamma)\mathrm{I}((m^1, m_2, \phi_1), (m^1, m_2, \phi_2), \gamma)\mathrm{I}(m^1, m_2, \phi_2 + \gamma(\phi_1 - \phi_2))$. On the other hand, by Theorem 5, we have $(x, y, \alpha_2 + \gamma(\alpha_1 - \alpha_2))\mathrm{I}((m^1, m_2, \alpha_x), (m^1, m_2, \alpha_y), \alpha_2 + \gamma(\alpha_1 - \alpha_2))\mathrm{I}(m^1, m_2, \alpha_y + (\alpha_2 + \gamma(\alpha_1 - \alpha_2))(\alpha_x - \alpha_y)) \equiv (m^1, m_2, \theta')$. But $\phi_2 + \gamma(\phi_1 - \phi_2) = \alpha_y + (\alpha_x - \alpha_y)(\gamma\alpha_1 + (1 - \gamma)\alpha_2) = \theta'$. Hence we have the required conclusion.

If $y \succ x$, then we put $x' = y$, $y' = x$, $\alpha' = 1 - \alpha$, and $\beta' = 1 - \beta$, and substitute them in the conclusion proved in the above case. Then, we have $((y, x, 1 - \alpha), (y, x, 1 - \beta), \gamma)\mathrm{I}(y, x, (1 - \beta) + \gamma(\beta - \alpha))$ Hence, by Lemma 3 and (a 5), we have $((x, y, \alpha), (x, y, \beta), \gamma)\mathrm{I}(x, y, \beta+\gamma(\alpha-\beta))$.

Theorem 11. Suppose $x, y\in T$, $x\succ y$. Then for any z such that $z\in T$ and $x\succ z\succ y$, there exists $\alpha(z)$ uniquely such that $z\mathrm{I}(x, y, \alpha(z))$, $1>\alpha(z)>0$, and if $x\succ z_1\succ z_2\succ y$ and $z_1, z_2\in T$, then, for any β, $1>\beta>0$, it holds that $\alpha((z_1, z_2, \beta))=\beta\alpha(z_1)+(1-\beta)\alpha(z_2)$.

Proof. By Theorem 7, $\alpha(z)$ exists uniquely for each z. By Theorem 8, we have $z_1\succ(z_1, z_2, \beta)\succ z_2$, so that $x\succ(z_1, z_2, \beta)\succ y$. Since $z_i\mathrm{I}(x, y, \alpha(z_i))$, $i=1,2$, by Theorem 10, we have $(z_1, z_2, \beta)\mathrm{I}(x, y, \alpha(z_2)+\beta(\alpha(z_1)-\alpha(z_2)))$. Hence, by Theorem 9, we have $\alpha((z_1, z_2, \beta))=\alpha((x, y, \alpha(z_2)+\beta(\alpha(z_1)-\alpha(z_2))))=\alpha(z_2)+\beta(\alpha(z_1)-\alpha(z_2))$.

Section V. The von Neumann-Morgenstern Theorem

Theorem 12. Suppose $x, y\in T$, $x\succ y$. Let $\alpha(z)$ be the function defined in Theorem 11. Consider any real-valued function $f(z)$ defined for all $z\in T$, $x\succ z\succ y$ such that (1) if $z_1\mathrm{I}z_2$, $x\succ z_i\succ y$, $i=1,2$, then $f(z_1)=f(z_2)$, (2) $f((z_1, z_2, \beta))=\beta f(z_1)+(1-\beta)f(z_2)$ for any z_i, $x\succ z_i\succ y$, $i=1,2$, $1>\beta>0$, and (3) if $x\succ z_1\succ z_2\succ y$, then $f(z_1)>f(z_2)$. Then, such a function $f(z)$ exists, and always takes the form $f(z)=c\alpha(z)+d$, where c and d are real numbers and $c>0$. Any such function $c\alpha(z)+d$ fulfils the properties (1) to (3).

Proof. By Theorems 9 and 11, $\alpha(z)$ fulfils the properties (1), (2) and (3) for $f(z)$. So does any function $c\alpha(z)+d$, $c>0$. Hence $f(z)$ exists. Let $f(z)$ be a function which fulfils (1) to (3). Take arbitrarily z_1 and z_2 such that $x\succ z_1\succ z_2\succ y$, and determine real values c and d so as to have $f_1=c\alpha_1+d$ and $f_2=c\alpha_2+d$,......(a), where $f_i=f(z_i)$ and $\alpha_i=\alpha(z_i)$, $i=1,2$. By Theorem 9, we have $\alpha_1>\alpha_2$, so that we have $c=(f_1-f_2)/(\alpha_1-\alpha_2)$ and $d=(\alpha_1 f_2-\alpha_2 f_1)/(\alpha_1-\alpha_2)$ to be finite definite. By the property (3) of $f(z)$, we have $c>0$.

Suppose, for some z_3, $x\succ z_3\succ y$, $f(z_3)\neq c\alpha_3+d$,......(b), where $\alpha_3=\alpha(z_3)$. If $z_3\succ z_1$, then, we have, by Theorem 7, that $z_2\mathrm{I}(z_3, z_1, \beta)$ for some unique β, $1>\beta>0$. Hence, by the property (2) of $f(z)$, we have $f_2=\beta f_3+(1-\beta)f_1$.......(c). But since, by Theorem 11, we have $\alpha_2=\alpha((z_3, z_1, \beta))=\beta\alpha_3+(1-\beta)\alpha_1$.......(d). Multiplying the terms on both sides of Eq. (d) by c and adding d to them, we have $c\,\alpha_2+d=\beta(c\,\alpha_3+d)+(1-\beta)(c\,\alpha_1+d)$, so that, by (a) and (b), noting $\beta\neq 0$, we have $c\,\alpha_2+d=f_2=\beta(c\,\alpha_3+d)+(1-\beta)f_1\neq\beta f_3+(1-\beta)f_1$, contradicting (c).

If $z_2\succ z_3$, then it is easy to have a contradiction to (c) by a similar reasoning.

Suppose, then, that $z_1\succ z_3\succ z_2$. By Theorem 7, we have $z_3\mathrm{I}(z_1, z_2, \beta)$ for some β. By the property (2) of $f(z)$, we have $f_3=\beta f_1+(1-\beta)f_2$. But by Theorem 11, we have $\alpha_3=\alpha((z_1, z_2, \beta))=\beta\alpha_1+(1-\beta)\alpha_2$, so that, by (a) and (b), we have $f_3\neq c\alpha_3+d=\beta(c\,\alpha_1+d)+(1-\beta)(c\,\alpha_2+d)=\beta f_1+(1-\beta)f_2$. This is a contradiction.

All this implies that, once given the values of $f(z)$ at any z_1, z_2, $x\succ z_1\succ z_2\succ y$, there must be some c and d as determined by (a), and once c and d are determined, of which c is positive, we have $f(z)=c\alpha(z)+d$ for all $z\in T$, $x\succ z\succ y$.

Corollary 1. Let x' and $y'\in T$ be $x'\succ x\succ y\mathrm{I}y'$, or $x'\mathrm{I}x\succ y\succ y'$, or $x'\succ x\succ y\succ y'$. If $f'(z)$ is any real-valued function defined for all $z\in T$, $x'\succ z\succ y'$ such that (1) if $z_1\mathrm{I}z_2$, then $f'(z_1)=f'(z_2)$, (2) $f'((z_1, z_2, \beta))=\beta\, f'(z_1)+(1-\beta)f'(z_2)$, $\forall\beta$, $1>\beta>0$, and (3) if $z_1\succ z_2$, then $f'(z_1)>f'(z_2)$, for any z_1, $z_2\in T$, $x'\succ z_i\succ y'$, $i=1,2$, then this function $f'(z)$ is uniquely determined up to a positive-sloped linear transformation. Also, any such function $f'(z)$ is an extension of some function $f(z)$ for $x\succ z\succ y$ defined in Theorem 12, in the sense that $f'(z)$ is defined for an extended domain $x'\succ z\succ y'$ of $x\succ z\succ y$ and $f'(z)$ takes the same values as $f(z)$ for all $z\in T$, $x\succ z\succ y$.

Proof. Let $f_i'(z)$, $i=1,2$, be functions fulfilling the properties (1) to (3). Let $\alpha'(z)$ denote the function defined for $x'\succ z\succ y'$ just as $\alpha(z)$ is defined for $x\succ z\succ y$. Then, by Theorem 12, we have $f_i'(z)=c_i\alpha'(z)+d_i$, and $c_i>0$, $i=1,2$. Hence $f_1'(z)=c_1(f_2'(z)-d_2)/c_2+d_1=(c_1/c_2)f_2'(z)-d_2(c_1/c_2)+d_1$, so that $f_1'(z)$ is a positive-sloped linear transformation of $f_2'(z)$.

Consider the function $f''(z)$, defined as $f''(z)=f'(z)$ for all $z\in T$, $x\succ z\succ y$, and for this domain only. Let $\alpha(z)$ denote the same function defined in Theorem 11. Then, since $f''(z)$ preserves all the properties (1) to (3) of $f'(z)$ for the narrower domain $x\succ z\succ y$, by Theorem 12, we have $f''(z)=c''\alpha(z)+d''$, and $c''>0$. Putting $f(z)=c\alpha(z)+d$, we have $f''(z)=c''((f(z)-d)/c)+d''$,......(i). Since $c>0$ and d are arbitrary in (i), we may take $c=c''$ and $d=d''$ in (i), and we have $f(z)=f''(z)$ for all $z\in T$, $x\succ z\succ y$. Thus, $f'(z)$ is an extension of this $f(z)$.

Corollary 2. Let $f(z)$ be a function defined in Theorem 12 for $x\succ z\succ y$. Suppose there are x', $y'\in T$ such that $x'\succ x$ and/or $y\succ y'$, and $x'\succsim x$ and $y\succsim y'$. Then there uniquely exists a function $f'(z)$ such that $f'(z)$ is an extension of $f(z)$ from $x\succ z\succ y$ to $x'\succ z\succ y'$, and $f'(z)$ fulfils the properties (1) to (3) in Corollary 1.

Proof. By assumption the function $f(z)$ is given. Define $f'(z)=c'\alpha'(z)+d'$ by $\alpha'(z)$ for the pair x' and y' so as to have $f'(z_i)=f(z_i)$ for some z_i, $i=1,2$, $x\succ z_1\succ z_2\succ y$. Then, by $f(z_1)>f(z_2)$, $c'>0$, and $f'(z)$ fulfils the properties (1) to (3) of Corollary 1. Now, define $f''(z)$ as in the proof of Corollary 1. Since $f(z)=c\alpha(z)+d$, we have, by the preserved properties of $f''(z)$, $f''(z)=c''\alpha(z)+d''$. Hence, $f''(z)=(c''/c)f(z)+(cd''-c''d)/c$, where $c>0$ by assumption. Here, since $f''(z_i)=f'(z_i)$, $i=1,2$, c'' and d'' must be such that $f''(z_i)=f(z_i)$ for the above z_i, $i=1,2$, $x\succ z_1\succ z_2\succ y$. We then have $f(z_i)=f''(z_i)=(c''/c)f(z_i)+(cd''-c''d)/c$, $i=1,2$, so that $(c-c'')f(z_1)=cd''-c''d=(c-c'')f(z_2)$. Therefore, since $f(z_1)\neq f(z_2)$, we have $c=c''$ and, by $c, c''>0$, $d=d''$. Thus, $f''(z)$ coincides with $f(z)$ for all $z\in T$, $x\succ z\succ y$. Obriously, the two values $f(z_i)$, $i=1,2$, uniquely determines (by $f'(z_i)=f(z_i)$, $i=1,2$) the function $f'(z)$ with the properties (1) to (3) in Corollary 1. Hence, the uniqueness of the extension of $f(z)$ to $x'\succ z\succ y'$.

Let us proceed to prove the von Neumann-Morgenstern theorem in each of the cases in which the set T does and does not contain extremums with respect to the preference ordering.

Theorem 13. There exists a function $f(z)$ defined for all $z \in T$ such that (1) if $x \mathrm{I} y$, $x, y \in T$, then $f(x)=f(y)$, (2) $f((x, y, \alpha))=\alpha f(x)+(1-\alpha)f(y)$ for any $x, y \in T$, $1>\alpha>0$, and (3) if $x \succ y$, $x, y \in T$, then $f(x)>f(y)$. Such a function $f(x)$ is uniquely determined up to a positive-sloped linear transformation.

(I) *The case without extremal prospects: Proof*

Let T be 'extremumless' with respect to the preference ordering, i. e., have no maximal or minimal prospects.

Existence. This assumption implies that for any $x, y \in T$, $x \succ y$, there are $x', y' \in T$ such that $x' \succ x \succ y \succ y'$. Then, choose $x, y, z_1, z_2 \in T$, $x \succ z_1 \succ z_2 \succ y$, and specify $f(z_i)$, $i=1, 2$ to determine c and d as in the proof of Theorem 12, thus determining $f(z)$ for $x \succ z \succ y$. Since, by assumption, $x' \succ x \succ y \succ y'$, we can extend $f(z)$ from $x \succ z \succ y$ to $x' \succ z \succ y'$, and then the values $f(x)$ and $f(y)$ are determined. (See Corollary 2.) Similarly, by the assumption of extremumlessness, the values $f(x')$ and $f(y')$ are alse determined by further extending the function.

Uniqueness. By Corollary 2, for any extension of the domain, the values of the extended function on the additional domain are uniquely determined. By the definition of the extension, the values of the extended function on the initial domain before the extension are preserved after it. Let $f(z)$ and $f'(z)$ be any two functions defined for all $z \in T$, fulfilling the properties (1) to (3). By Corollary 1 the one must be a positively sloped linear transformation of the other, when they are viewed as functions defined only on the same subset $x \succ z \succ y$ of T. This subset is arbitrary, and hence the uniqueness in this case is verified.

(II) *The case with extremal prospects: Proof*

Let T have some maximal and/or minimal prospects.

Existence. (i) Suppose there are some $x', y' \in T$, $x' \succsim z \succsim y'$ for any $z \in T$. Consider the function $\alpha'(z)$ for $x' \succ z \succ y'$ in Theorem 11.

Extend the function $\alpha'(z)$ to $x' \succsim z \succsim y'$ by defining $\alpha'(x')=1$ and $\alpha'(y')=0$. Then, (i-1) $(x', y', \beta)=(x', y', \beta.1+(1-\beta).0)=(x', y', \beta\alpha'(x')+(1-\beta)\alpha'(y'))$, (i-2) $(x', (x', y', \alpha), \beta)\mathrm{I}(x', y', \alpha+\beta-\alpha\beta)=(x', y', \beta.1+(1-\beta)\alpha)=(x', y', \beta\,\alpha'(x')+(1-\beta)\alpha'(z'))$, and (i-3) $((x', y', \alpha), y', \beta)\mathrm{I}(x', y', \alpha\beta)=(x', y', \beta.\alpha+(1-\beta).0)=(x', y', \beta\alpha'(z')+(1-\beta)\alpha'(y'))$, where $z'=(x', y', \alpha)$ Hence, we have $\alpha'((x, y, \beta))=\beta\alpha'(x)+(1-\beta)\alpha'(y)$ for any $x, y, x' \succsim x \succsim y \succsim y'$, $1>\beta>0$......(1). Define $\alpha(z)=1, 0$ for z such that $z\mathrm{I}x'$ and $\mathrm{I}y'$, resp.

Now, define $f'(z)$ for $x' \succsim z \succsim y'$ by $f'(z)=c\,\alpha'(z)+d$. Then, $f'(x')=c+d$ and $f'(y')=d$, and by Eq. (1), we have $f'((x, y, \beta))=c(\beta\alpha'(x)+(1-\beta)\alpha'(y))+d=\beta(c\alpha'(x)+d)+(1-\beta)(c\alpha'(y)+d)=\beta f'(x)+(1-\beta)f'(y)$ for any $x, y, x' \succsim x \succsim y \succsim y'$. $f'(z)$ clearly fulfils the properties (1) and (3).

(ii) Suppose there is a maximal element $x' \in T$, i.e., such that $x' \succsim x$ for any $x \in T$, but there is no minimal element to T. Let the interval for which the initial $f(z)$ before any extention is defined be $x \succ z \succ y$. Choose any y', such that $y' \prec y$. Define $\alpha'(x)$, as in Subcase (i) above, for $x' \succsim z \succsim y'$. By use of this $\alpha'(x)$, starting from the interval $x \succ z \succ y$, the function $f(z)$ can be extended to $x' \succsim x \succsim y'$ as in the proof of Corollary 2. Since y' is arbitrary, it can be extended all over T.

(iii) In the case where there is a minimal element but not a maximal element to T, the proof is similar to that of (ii).

Uniqueness. Let $f(z)$ and $f'(z)$ be functions defined for all $z \in T$ which fulfil the properties (1) to (3).

(i) Suppose there are $x', y' \in T, x' \succsim z \succsim y'$ for all $z \in T$. Then, for $x' \succ z \succ y'$, $f(z)=c\,\alpha(z)+d$ and $f'(z)=c'\,\alpha(z)+d'$, where $\alpha(z)$ is the function defined in Theorem 11 for $x' \succ z \succ y'$. Also, by Corollary 2 of Theorem 12, for $x' \succ z \succ y'$, $f'(z)=af(z)+b$, where $a>0$ and b are constant. Let us show that (i)-1 : we must define $f(x')=c+d$, $f'(x')=c'+d'$, and $f(y')=d$, $f'(y')=d'$, and that (i)-2 : $f'(x')=af(x')+b$ and $f'(y')=af(y')+b$.

(i)-1 : Suppose the value $f(x')$ we define is not equal to $c+d$. Then, by the property (2) of $f(z)$ for the domain $x' \succsim z \succ y'$, we have for $z=(x,' y', \alpha)$, $f((x', z, \beta))=\beta f(x')+(1-\beta)f(z) \neq \beta(c+d)+(1-\beta)(c\alpha+d)=c(\beta+(1-\beta)\alpha)+d=c\alpha((x', z, \beta))+d$ by $(x', z, \beta)\mathrm{I}(x', y', \alpha+\beta-\alpha\beta)$ and Theorem 9. Since $x' \succ (x', z, \beta) \succ y'$, this equals to $f((x', z, \beta))$, a contradiction. Hence, we must define $f(x')=c+d$. Similarly $f'(x')=c'+d'$. Also, by the similar reasoning, we must define $f(y')=d$ and $f'(y')=d'$. We define $f(z)=c+d$ for $z\mathrm{I}x'$, etc.

(i)-2 : Let z_1, $z_2 \in T$ be $x' \succ z_1 \succ z_2 \succ y'$, and f_i, f_i' denote $f(z_i)$, $f'(z_i)$, $i=1$, 2, resp. Then, we have $f_1'=af_1+b$ and $f_2'=af_2+b$...... ...(a). Suppose $f'(x') \neq af(x')+b$.........(b). Then, since there uniquely exists γ, $z_1\mathrm{I}(x', z_2, \gamma)$, $1>\gamma>0$, we have, by the property (2) of $f'(z)$, $f_1'=\gamma f'(x')+(1-\gamma)f_2'$.........(c). But by the property (2) of $f(z)$, we also have $f_1=f((x', z_2, \gamma))=\gamma f(x')+(1-\gamma)f_2$.........(d). From Eq. (d), we have $af_1+b=\gamma(af(x')+b)+(1-\gamma)(af_2+b)$, so that, by (a) and (b), noting $\gamma \neq 0$, we have $f_1'=af_1+b=\gamma(af(x')+b)+(1-\gamma)f_2' \neq \gamma f'(x')+(1-\gamma)f_2'$, contradicting to (c). Also, if we suppose $f'(y') \neq af(y')+b$, we have a contradiction by the similar reasoning. Hence, we have $f'(z)=af(z)+b$ for all $z, x' \succsim z \succsim y'$, i. e., for all $z \in T$.

(ii) Suppose there is a maximal (minimal) element x' of T but no minimal (maximal, resp.) element y' of T. Let us deal with with only the first case, since the other case can be similarly treated. Let $y' \in T$ be $x' \succ y'$. Then, for $x' \succ z \succ y'$, $f(z)=c\,\alpha(z)+d$, $f'(z)=c'\alpha(z)+d'$, and, by Corollary 2, $f'(z)=af(z)+b$ for all $z, x' \succ z \succ y'$, where $\alpha(z)$ is the function pertinent to x' and y'. By exactly the same reasonings as those in Subcase (i) just above, it is verified that (1) we must define $f(x')=c+d$ and $f'(x')=c'+d'$ and that (2) we then have $f'(x')=af(x')+b$. Thus, we have $f'(z)=af(z)+b$ for all $z, x' \succsim z \succ y'$. Since y' is arbitrary, we have $f'(z)=af(z)+b$ for all $z \in T$.

Conclusion

If we take all the axioms (a 1) to (a 7) for granted, (1) Theorem 12 and its two corollaries ensure existence of a cardinal utility function defined all over *the set of sure prospects* S, which is unique up to a positive-sloped linear transformation, so that convexity or concavity, if definable, at any point $x \in S$ would be preserved, regardless of any such transformation. Certainly, they ensure existence of a cardinal utility function defined all over *the set of general prospects* T, but in nothing but that lies the point essentially relevant to the classical and fundamental economic problem of cardinal utility.

(2) They also enable us to carry on recursively the following separating operations with respect to the utility function thus defined on T:

Let $x=(((a, b, \alpha_1), c, \alpha_2), (d, (e, f, \alpha_3), \alpha_4), \alpha_5)$, e.g. where a, b, etc. belong to S, and all α_i, s belong to the open interval (0, 1). Let $u(z)$ be a utility function which has been constructed on the basis of our axioms (a 1) to (a 7) so as to fulfil the properties (1) to (3) of Theorem 12.

Then, by the property (2), we have

$$
\begin{aligned}
u(x) &= \alpha_5 u(((a, b, \alpha_1), c, \alpha_2)) + (1-\alpha_5) u((d, (e, f, \alpha_3), \alpha_4)) \\
&= \cdots\cdots\cdots\cdots \\
&= \alpha_5(\alpha_2(\alpha_1 u(a) + (1-\alpha_1) u(b)) + (1-\alpha_2) u(c)) \\
&\quad + (1-\alpha_5)(\alpha_4 u(d) + (1-\alpha_4)(\alpha_3 u(e) + (1-\alpha_3) u(f))) \\
&= \sum_{k=1}^{5} \beta_k u(x_k),
\end{aligned}
$$

where $x_k = a, b, c, d, e$, and f for $k=1, 2, \ldots\ldots, 5$, resp., $\sum_{k=1}^{5} \beta_k = 1$, and $\beta_k > 0$ for $k=1, 2, \ldots\ldots, 5$.

If $T/I - S/I \neq \phi$, then it is only at the stage of Theorem 10 (which is a basis of Theorem 12) that such a calculation as the above example shows becomes ensured to be possible at least in the logical context of the analysis of this paper. In fact, even the first equality of the above calculation cannot be logically justified before arriving at Theorem 10, for, in this general case of $T/I-S/I\neq\phi$, the multi-stage lotteries $((a, b, \alpha_1), c, \alpha_2)$ and $(d, (e, f, \alpha_3), \alpha_4)$ *may* not have any corresponding elements of S which are equivalent in utility to themselves, and, if so, such a proposition as Theorem 12 will be required, even after Theorem 5 has been proved, which reads only in terms of elements of S.

The calculation (1) in Introduction will be verified as follows: Rewrite the argument of the term on the left-hand side of (1) to the form of a multi-stage (binary) lottery like that of x above. By the calculation similar to that applied to x in the above, we will get the weighted average expression. It will be easy to show that the resulting weights in the expression coincide with those probabilitles in the original argument of the left-hand side of (1).

(3) Theorem 13 indicates that under the axioms the cardinal utility function exists (uniquely in that sense) even when the set S *is extremumless* with respect to the relation '$\succ$.' (The range of the function may or may not be unbounded.)[1]

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1) R. C. Merton's remark that 'the [von Neumann-Morgenstern] original axioms require that [the cardinal utility function] is bounded' (footnote 2, P. 602, [11]) might seem misleading. More accurately, in order to exclude the possibility of an infinite sum of utility of the St. Petersburg game, it has to be bounded.

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