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Local stability conditions for two types of monetary models with recursive utility *†

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Abstract

This paper explores local stability conditions for money-in-utility-function (MIUF) and transaction-costs (TC) models with recursive utility. A monetary variant of the Brock–Gale condition provides a theoretical justification of the comparative statics analysis. One of sufficient conditions for local stability is increasing marginal impatience (IMI) in consumption and money. However, this does not deny the possibility of decreasing marginal impatience (DMI). The local stability with DMI is more plausible as the elasticity of the money demand function becomes lower and the curvatures of the production, the felicity, the discount rate, and the transaction-costs functions become higher.

JEL code: O42
Keywords: recursive utility, money-in-utility-function model, transaction-costs model, decreasing marginal impatience.

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1 Introduction

Several papers have analyzed the relationship between inflation and growth in the money-in-utility-function (MIUF) models (Sidrauski, 1969) and transaction-costs (TC) models (Saving, 1971). Most of these papers, such as Wang and Yip (1992) and Zhang (2000), used a utility function with a constant time preference and an endogenous labor decision and showed that higher growth rates of money lower capital in the long-run. On the other hand, Chen et al. (2008) considered a utility function with an endogenous time preference, or a recursive utility (Koopmans, 1960). Without labor-leisure trade-offs, they discovered that the MIUF (resp. TC) model predicts a positive correlation between inflation and growth (Tobin, 1965) in the case of increasing marginal impatience (IMI) with respect to real balances (resp. consumption).[1] Similarly, decreasing marginal impatience (DMI) leads to a reverse Tobin effect. They concluded that a reverse Tobin effect was more plausible because there were empirical results supporting DMI in real balances (Becker and Mulligan, 1997, among others).

Chen et al. (2008) made a valuable contribution to the examination of the relationship between inflation and growth, but they conducted only a comparative static analysis in a steady state, leaving the transitional dynamics unexplored. The steady-state equilibrium might be unstable, or might

[1] Hayakawa (1995) showed that capital and money are invariant to monetary growth in the case of the MIUF model with recursive utility and perfect complementary of consumption and money.

generate multiple equilibriums. Investigating the stability properties in the neighborhood of the steady state involves a linearized dynamic system with a Jacobian matrix. Even in the case of comparative statics, checking the sign of the determinant of the matrix is required. However, Chen et al. simply assumed that the determinant is negative. The major reason for this assumption may have been that the system was a four-dimensional dynamics and, hence, was too complicated to be analyzed.

Our paper reinforces the comparative statics of Chen et al. (2008) by examining the stability properties of the dynamic system of the MIUF and TC models in the neighborhood of the steady state. We undertake this analysis using the method proposed by Obstfeld (1990) and Chang (1994). They considered a one-sector, nonmonetary, optimal growth model with recursive utility. Exploiting the facts that the value function is the same as a costate variable of the dynamic system on the optimal path and that the derivative of the value function is the other costate variable, they succeeded in reducing the dimensions of the dynamic system from three to two. Similarly, our paper reduces the dimensions of our two monetary models from four to three.

Fischer (1979) and Calvo (1979) investigated the sufficient conditions for the local stability of the MIUF model with a constant time preference. The dynamic system of the monetary model has three dimensions, with two jump variables, requiring one negative root and two positive roots in the characteristic function. To establish only one negative root in the dynamic system, the determinant (the products of the three roots) should be negative, and the trace (the sum of the roots) should be positive. For local stability, Fischer (1979) assumed that the instantaneous utility function is concave with respect to consumption and real balances and that consumption and real balances are normal goods. Instead of the latter conditions, Calvo (1979) proposed a sufficient but simple condition, in which both goods are gross substitutes.

Since Uzawa (1968), a great deal of literature has been devoted to analyzing models with recursive utility. Few studies deal with the stability conditions of monetary models, although Uzawa is one exception. In a different framework from our models, he insisted that impatience should be marginally increasing in felicity.[2] Another exception is Epstein and Hynes (1983). Their MIUF model assumed that the instantaneous utility function is a negative constant, that marginal impatience is increasing and concave in consumption and real balances, and that both goods are normal. Under these assumptions, Epstein and Hynes claimed that the dynamic system is locally stable.

Even in the case of non-monetary models with recursive utility, most studies have considered only the stability case by assuming IMI in consumption or in felicity from the beginning. Das (2003) investigated the condition for stability in the case of DMI. Her condition implies that local stability needed a variant of the Brock–Gale (BG) condition, in which an increase in

[2]He also assumed that felicity takes a positive value and that the discounting rate function is convex.

the discount rate dominates the increase in the marginal product of capital at the steady state (Brock and Gale, 1969). Although Chen et al. (2008) assumed this BG condition, called the Correspondence Principle, in their paper, they used it simply to discuss the unique existence of the steady state. Furthermore, the BG condition that Chen et al. assumed ignored the effect of money. If a monetary variant of the BG condition is established correctly, we conjecture that this condition will play an essential role in determining a negative determinant of the matrix.

Our paper offers local stability conditions for two types of monetary models with recursive utility, in which the determinant of the Jacobian matrix is negative and the trace is positive. Our finding is as follows. A negative determinant of the matrix calls for a decreasing money demand function in the nominal interest rate, a positive intertemporal substitution, and a monetary variant of the BG condition in both models. A sufficient condition for a positive trace of the matrix is increasing money demand in consumption and IMI in consumption for both models.

In particular, for the MIUF model, a sufficient condition for a monetary BG condition or a negative determinant of the Jacobian matrix is IMI in money and an increasing money demand in consumption and capital. On the other hand, a sufficient condition for a positive trace of the matrix involves IMI in consumption and increasing money demand in consumption, but decreasing money demand in capital. Thus, for local stability, impatience should be marginally increasing both in consumption and money, and the money demand function should be increasing in consumption and independent of capital. Such local stability conditions regarding the MIUF model encompass those of Uzawa (1968), Fischer (1979), Calvo (1979), and Epstein and Hynes (1983).

However, this does not deny the possibility of DMI. The local stability with DMI is more plausible as: (i) the production function becomes more concave and the money demand function becomes less elastic in both models; (ii) the felicity taking a negative (resp. positive) value becomes more concave, and the discounting rate becomes more concave (resp. convex) in the MIUF model; and (iii) the transaction technology becomes more concave in the TC model.

The remainder of the paper is organized as follows. Section 2 examines the MIUF model and Section 3 analyzes the TC model. Each section describes the model economy and investigates the local stability around the steady state. Section 4 concludes the paper.

2 The MIUF model

This section considers the local stability of the MIUF model, in which money or real balances directly enters the felicity function. The model structure as well as the notations closely follow Chen et al. (2008) to facilitate a comparison between their and our results. Our assumption set is a little different from theirs because we focus our attention on the neighborhood of the steady state by assuming its existence and we allow the felicity function to take a positive or a negative value. The first subsection describes the model economy and the definition of the steady state. The next subsection examines the local stability. Starting with the minimal assumptions, we add the assumptions needed for comparative statics and for dynamic properties.

2.1 The model economy

In the economy, there exist infinitely lived consumers and the government. All consumers have the same preference and are represented as a single agent. We assume no population growth and exogenous technological progress.

The representative agent with perfect foresight solves the following problem:

$$\max_{\{c_t, m_t\}} \int_0^\infty e^{-\Delta_t} u(c_t, m_t) dt, \quad \text{s.t.}$$

$$\dot{a}_t = f(k_t) - \pi_t m_t + v_t - c_t, \tag{1}$$

$$\dot{\Delta}_t = \rho(c_t, m_t), \tag{2}$$

$k_0 > 0$ and $m_0 > 0$,[3] where c_t is consumption, m_t is real money balances, k_t is capital, $a_t = k_t + m_t$ is assets, π_t is the inflation rate, v_t is lump-sum government transfers, f represents the production function, u represents the felicity function, ρ represents the subjective discount rate function, and

[3]Initial positive money stock is required for the finite growth rate of the money supply by the government at the next continuous period.

Δ represents the cumulative subjective discount rate function. A higher ρ indicates an agent with higher marginal impatience.[4] This functional form contains the case of Uzawa (1968) by setting $\rho = \tilde{\rho}(u(c, m))$. We call such a discount rate function a Uzawa-type function.

We assume that u, ρ, and f are twice continuously differentiable. We assume that $u_c \geq 0$, $u_m \geq 0$, $u_{cc} \leq 0$, $u_{mm} \leq 0$, $\rho > 0$ for all $c > 0$ and $m > 0$, and $f > 0$ for all $k > 0$. These are quite standard assumptions. More assumptions regarding u, ρ, and f are imposed after the steady state is defined.

The level of felicity u is often assumed to be negative in the field of literature that includes Epstein and Hynes (1986) and Obstfeld (1990). The class of felicity function taking a negative value covers the Epstein–Hynes function ($u = -1$) and the famous CRRA functional form $(c^{1-\alpha}m^{\alpha})^{1-\sigma}/(1-\sigma)$ for $\sigma > 1$ and $0 < \alpha < 1$. Such negativeness is a sufficient condition for the concavity of the Hamiltonian, defined later. On the other hand, some authors, including Uzawa (1968), Das (2003), and, implicitly, Chen et al. (2008), assume that $u > 0$. This paper allows the level of felicity to be either positive or negative, but it cannot equal zero ($u \neq 0$) for all c.

We do not impose any restrictions on the first degree of derivatives of ρ. Whereas $\rho_c > 0$ and $\rho_m > 0$ are referred to as IMI in consumption and money, $\rho_c < 0$ and $\rho_m < 0$ are referred to as DMI. Most authors, except

[4]Some authors, including Chen et al. (2008), use impatience instead of marginal impatience.

for Das (2003), have assumed IMI in a real economy for stability purposes, although they have admitted that IMI is not consistent with several works in the empirical literature (Becker and Mulligan, 1997, for example). According to Chen et al. (2008), a reverse Tobin effect, a negative relationship between inflation and economic growth, emerges when $\rho_m < 0$ in the MIUF model, but they assumed the steady state to be locally stable. This paper investigates what other conditions are needed in the case of IMI, or whether DMI is justified from a theoretical viewpoint, and thus starts with no restrictions on ρ_c and ρ_m.

To solve the above problem, we consider the following present value Hamiltonian:

$$H = e^{-\Delta}\{u(c,m) + \lambda(f(k) - \pi m + v - c) - \phi\rho(c,m) + \psi(a - m - k)\}, \quad (3)$$

where ψ denotes the Lagrange multiplier for $a = m + k$, and λ and ϕ denote the costate variables associated with (1) and (2), respectively. Applying the Pontryagin maximum principle leads to the following first-order necessary conditions:

$$u_c - \lambda - \phi\rho_c = 0 \quad (4)$$

$$u_m - \lambda(f_k + \pi) - \phi\rho_m = 0 \quad (5)$$

$$\dot{\lambda} = \lambda(\rho - f_k) \quad (6)$$

$$\dot{\phi} = -u + \phi\rho, \quad (7)$$

(1), (2), $a = m + k$, the initial conditions $k_0 > 0$, and $m_0 > 0$, and the transversality conditions $\lim \lambda_t a_t e^{-\Delta_t} = 0$ and $\lim \phi_t \Delta_t e^{-\Delta_t} = 0$. Since $a = m + k$ is binding, $\psi = \lambda f'(k) > 0$.

For a sufficient condition for the maximization problem, we should assume that the Hamiltonian (3) is concave with respect to c, m, k, a, and Δ for any multipliers λ, ϕ, and ψ in a strict sense. One of the sufficient conditions is that $e^{-\Delta}u(c,m)$, $f(k)$, and that $\rho(c,m)$ are concave when the costate variables in the Hamiltonian are set to $\tilde{\phi} = -e^{-\Delta}\phi$ and $\tilde{\lambda} = e^{-\Delta}\lambda$. The concavity of $e^{-\Delta}u(c,m)$ holds if $u < 0$, $u_{cc} < 0$, $u_{mm} < 0$, $u_{cc}u_{mm} - u_{cm}^2 > 0$, and $u(u_{cc}u_{mm} - u_{cm}^2) + u_c(u_m u_{cm} - u_{mm}u_c) + u_m(u_c u_{cm} - u_{cc}u_m) > 0$ for all c and m, excluding the possibility that $u > 0$. However, even if these are not satisfied, the indirect utility function might be concave. Assuming the indirect utility or the value function to be concave, Chang (1994) adopted a dynamic programming approach and allowed the felicity to take positive and negative values. This paper does not start with the sufficient condition from the Hamiltonian concavity. We return to discussing that issue after we have determined all the required assumptions for local stability.

The solution to (7) with $\lim \phi \Delta e^{-\Delta} = 0$ is:

$$\phi(t) = \int_t^{\infty} e^{-\int_t^v \rho(c_\tau, m_\tau) d\tau} u(c_v, m_v) dv,$$

which is the lifetime utility from the period t. The value of ϕ takes a negative (resp. positive) value when $u <$ (resp. $>$) 0. As Obstfeld (1990) and Chang

(1994) have shown, $\phi(t)$ on the optimal path is the same as the solution to the following continuous version of the Bellman equation:

$$V(a) = \max_{c_t, m_t, 0 \leq t \leq \delta t} \left\{ \int_0^{\delta t} e^{-\Delta_t} u(c_t, m_t) dt + e^{-\Delta_{\delta t}} V(a + \delta a) \right\},$$

with the budget constraint (1) and $a = m + k$ for a small $\delta t > 0$. When $\delta t \to 0$, the Bellman equation is:

$$0 = \max_{c,m} \{u(c, m) - \rho(c, m)V(a) + \dot{a}V'(a)\}.$$

subject to (1) and $a = m + k$. The first-order condition with respect to c leads to $V'(a) = u_c - \rho_c V(a)$. The marginal utility in consumption depends on marginal felicity as well as future lifetime utility through an endogenous time preference. Clearly, $\lambda = u_c - \rho_c \phi = V'(a)$. We assume that the lifetime utility is increasing in assets, and that the shadow price of assets is positive.

We define the money demand function used for investigating the local stability of the dynamic system in the next subsection. Let $R_t = f_k + \pi_t$ represent nominal interest rates, the cost of holding money. We implicitly define the money demand function $\varphi(c, k, R)$[5] as

$$R_t = \frac{u_m(c, \varphi) - V(\varphi + k)\rho_m(c, \varphi)}{u_c(c, \varphi) - V(\varphi + k)\rho_c(c, \varphi)}. \tag{8}$$

[5]Here, a consumer behaves as a price taker and, accordingly, money demand is a function of c, k, and R even though R is a function of k in equilibrium. In addition, $\varphi(c, k, R)$ is more intuitive to explain than $\tilde{\varphi}(c, k, \pi) = \varphi(c, k, f_k + \mu)$.

The money demand function φ depends on nominal interest rates, consumption and capital. In the case of a constant time preference or of the Uzawa-type discount rate, the left side of (8) is simplified to $u_m(c,\varphi)/u_c(c,\varphi)$ and, therefore, the term of capital is not augmented in the money demand function φ.

Now we return to the remaining model components: the government and equilibrium conditions. The government prints money at a constant rate μ and runs a balanced budget by transferring seigniorage revenues to the consumers: $v_t = \mu m_t$. In equilibrium, the money and the goods markets are clear:

$$\dot{m} = (\mu - \pi)m \tag{9}$$

$$\dot{k} = f(k) - c. \tag{10}$$

We define a steady state $(c^*, m^*, k^*, \lambda^*, \phi^*, \pi^*)$ when the variables satisfy (4), (5), (6), (7), (9), (10), and $\dot{\lambda} = \dot{\phi} = \dot{m} = \dot{k} = 0$.

We assume that there exists a steady state such that $c^* > 0$, $m^* > 0$, $k^* > 0$, $\lambda^* = u_c - \rho_c\phi^* = V'(m^*+k^*) > 0$, $\phi^* = u(c^*, m^*)/\rho(c^*, m^*) = V(m^*+k^*)$, and $\pi^* = \mu$. We assume that both real and nominal rates of interests are positive: $f_k > 0$ and $\mu + f_k > 0$. To establish (8) with $m^* > 0$, $\lambda > 0$, and $R > 0$, we have to assume the positive marginal utility in real balances: $u_m - \rho_m\phi^* > 0$. The positive marginal utility in consumption and real balances is written as $u_c - u\rho_c/\rho > 0$ and $u_m - u\rho_m/\rho > 0$ for a steady state.

Both conditions hold automatically when $u < 0$, $\rho_c \geq 0$, and $\rho_m \geq 0$.

Furthermore, we assume that $f_{kk} < 0$ and $u_{cc} - u\rho_{cc}/\rho < 0$ in a steady state. The former implies that the marginal productivity of capital is decreasing. The latter is used for a positive intertemporal elasticity of substitution. Similarly, we assume that $u_{mm} - u\rho_{mm}/\rho < 0$.

Chen et al. (2008) assumed that $u_c/u - \rho_c/\rho > 0$ and $u_{cc}/u - \rho_{cc}/\rho < 0$ for a steady state because they assumed implicitly that $u > 0$. When $u < 0$, the inequalities should be the opposite. In addition, Chen et al. assumed that the curvature of the felicity is larger than that of the discount rate with respect to consumption. It should be noted that our assumption that $u_{cc} - u\rho_{cc}/\rho < 0$ always holds when $u < 0$, $u_{cc} < 0$, and $\rho_{cc} < 0$.

We make an additional assumption in regard to the money demand function φ. We assume that the money demand function is decreasing in the nominal interest rates for a steady state. That is, a higher cost of holding money R_t reduces money demand. Let the right-hand side of (8) be $L = (u_m - V\rho_m)/(u_c - V\rho_c)$. The total differential is $dR = L_m dm + L_c dc + L_k dk$, where L_x for $x = c, m, k$, is shown in (32), (33), (34) of Appendix A. The implicit function theorem indicates that $\varphi_R = 1/L_m$, $\varphi_c = -L_c/L_m$ and $\varphi_k = -L_k/L_m$. A decreasing money demand in nominal interest rates ($\varphi_R < 0$) is equivalent to $L_m < 0$ for a steady state. We do not make any assumptions regarding the signs of φ_c and φ_k (L_c and L_k) at this stage.

Before closing this subsection, we summarize the assumptions on the felicity u, the discount rate ρ, the production technology f, and the money demand φ that we have made so far.

1. $u \neq 0$, $u_c \geq 0$, $u_m \geq 0$, $u_{cc} \leq 0$, $u_{mm} \leq 0$ and $\rho > 0$ for all $c, m > 0$ and $f > 0$ for all $k > 0$.

2. $f_k > 0$, $\mu + f_k > 0$, $f_{kk} < 0$, $u_c - u\rho_c/\rho > 0$, $u_m - u\rho_m/\rho > 0$, $u_{cc} - u\rho_{cc}/\rho < 0$, and $u_{mm} - u\rho_{mm}/\rho < 0$ for a steady state.

3. $\varphi_R < 0$ or equivalently $L_m < 0$ for a steady state.

It should be noted that ρ_c and ρ_m may take negative values. Starting with these assumptions, we add the required conditions for the local stability of the dynamic system in the next subsection.

2.2 Local stability conditions

This subsection examines a local stability condition. For this purpose, we approximate the dynamic system around the steady state using the Jacobian matrix and then we reproduce the result of Chen et al. (2008) in our three-dimensional system to confirm that the determinant of the Jacobian matrix should be negative for justifying the comparative statics. Then, we examine the local stability of the dynamic system by considering the determinant and the trace of the matrix separately, which provides us with two propositions. Combining the two propositions yields our main results in this section as two theorems. The first theorem uses the money demand function, which is characterized by preferences. After examining the independence of money demand from capital, we offer the second theorem using only preferences and technology and a corollary. Finally, we show a functional example satisfying local stability as well as the concavity of the Hamiltonian.

Linearizing the dynamic system around the steady state leads to:

$$\begin{bmatrix} \dot{c} \\ \dot{m} \\ \dot{k} \end{bmatrix} = J_{MIUF} \begin{bmatrix} c - c^* \\ m - m^* \\ k - k^* \end{bmatrix}$$

where J_{MIUF} is the Jacobian matrix of the dynamic system, specified in (35) of Appendix A.

Before analyzing the local stability conditions, we mention the comparative statics conducted by Chen et al. (2008). The final part of Appendix A provides the following results:

$$\begin{bmatrix} dc^*/d\mu \\ dm^*/d\mu \\ dk^*/d\mu \end{bmatrix} = \frac{c^* m^*}{\theta \det(J_{MIUF})} \begin{bmatrix} -\rho_m f_k \\ -(f_{kk} - \rho_c f_k) \\ -\rho_m \end{bmatrix} \tag{11}$$

where $\theta = -(u_{cc} - \rho_{cc} V)c^*/(u_c - \rho_c V)$ is the intertemporal elasticity of substitution. By assumption, $\theta > 0$. The correlation between capital stocks and inflation depends only on the sign of ρ_m. That is, as long as $\det(J_{MIUF})$ is negative, $dk^*/d\mu >$ (resp. $<$) 0 if $\rho_m >$ (resp. $<$) 0. Our results are the same as those provided in Table 1 in Chen et al., although they assume that the Jacobian matrix of the four-dimensional dynamic system has a negative determinant.

Now, we start investigating the dynamic system. In the MIUF model, consumption, c, and real money balances, m, are jump variables; the consumer can choose any values of c and m. Therefore, we require only one negative root in the three-dimensional system for stability in terms of economic theory. As sufficient conditions, the determinant $\det(J_{MIUF})$ should be negative and the trace $\mathrm{tr}(J_{MIUF})$ should be positive. In what follows, we examine conditions for $\det(J_{MIUF}) < 0$ and $\mathrm{tr}(J_{MIUF}) > 0$ separately, and then combine them to provide the main results in this section in the form of two theorems.

First, we present conditions for $\det(J_{MIUF}) < 0$. With some algebra (see Appendix A):

$$\det(J_{MIUF}) = \frac{c^*m^*}{\theta}\{\rho_m(f_{kk} - L_k - f_kL_c) + L_m(f_k\rho_c - f_{kk})\}, \tag{12}$$

where L_x for $x = c, m, k$, is defined in the previous subsection and each concrete expression is given in Appendix A. Because $c^* > 0$, $m^* > 0$, $k^* > 0$, $\theta > 0$ and $L_m < 0$ by assumption, we can say that $\det(J_{MIUF}) < 0$ if:

$$\rho_m(f_{kk} - L_k - f_kL_c)/L_m + f_k\rho_c - f_{kk} > 0. \tag{13}$$

This condition is interpreted as a monetary version of the BG condition (Brock and Gale, 1969), in which the increase in the discount rate dominates the increase in the marginal product of capital ($f_{kk} < 0$) for a steady state. Chen et al. (2008) assumed that $f_k \rho_c - f_{kk} \geq 0$[6] and called it the CP condition. As $f_k = dc/dk$ in a steady state, the CP condition indicates the dominance of the increase in the discount rate over the increase in the marginal product of capital for a steady state, ignoring the effect of money on the degree of impatience. As a monetary version of the CP condition or the BG condition, we propose the condition (13) instead.

To justify our proposal, we first consider the meaning of $(f_{kk} - L_k - f_k L_c)/L_m$ in (13). Remember that the money demand function $\varphi(R, c, k)$ is implicitly defined as (8) and the total differential of $L = (u_m - \rho_m V(k + m))/(u_c - \rho_c V(k+m))$ is $dR = L_m dm + L_c dc + L_k dk$. Then, using the implicit function theorem, we can show that:

$$\frac{dm}{dk} = \varphi_R \frac{dR}{dk} + \varphi_c \frac{dc}{dk} + \varphi_k = \frac{1}{L_m} \left\{ \frac{dR}{dk} - L_c \frac{dc}{dk} - L_k \right\} = \frac{f_{kk} - L_k - f_k L_c}{L_m}.$$

Note that $R = f_k + \mu$ and $dR/dk = f_{kk}$ at a steady state. Thus, $(f_{kk} - L_k - f_k L_c)/L_m$ indicates the degree to which capital increases real balances under a steady-state equilibrium. Adding the direct effect of capital on consumption, dc/dk, we can obtain a monetary version of the BG condition, which is $\rho_m dm/dk + \rho_c dc/dk - f_{kk} > 0$. As $c^* > 0$, $m^* > 0$, $k^* > 0$, $\theta > 0$ and $\varphi_R < 0$ by assumption, the BG condition is equivalent to a negative determinant of the matrix.

[6] As discussed in the introduction, Chen et al. (2008) used this assumption simply to prove the unique existence of the steady state under the special case of $\rho_m = 0$.

The above discussion can be summarized as the following proposition:

Proposition 1. *Suppose the assumptions in the previous subsection hold. If the BG condition (13) holds, then* $\det(J_{MIUF}) < 0$.

Although the BG condition is satisfied when $\rho_m \geq 0$, $\rho_c \geq 0$, $L_k \geq 0$ and $L_c \geq 0$, it does not eliminate the possibility of a negative ρ_m for a steady state. A sufficiently high curvature of f and a sufficiently inelastic[7] demand function with respect to c, k, and R may lead to a reverse Tobin effect. As long as the BG condition holds, the comparative static analysis by Chen et al. (2008) is justified.

Next, we provide the conditions for $\mathrm{tr}(J_{MIUF}) > 0$. In Appendix A, we can show that:

$$\begin{aligned} \mathrm{tr}(J_{MIUF}) &= \frac{c^* m^*}{\theta}\rho_c L_c - m^* L_k + f_k \\ &\quad + m^* \frac{(u_{cm} - u\rho_{cm}/\rho)^2 - (u_{cc} - u\rho_{cc}/\rho)(u_{mm} - u\rho_{mm}/\rho)}{(u_c - u\rho_c/\rho)(u_{cc} - u\rho_{cc}/\rho)} \end{aligned} \tag{14}$$

for a steady state. Thus, if:

$$\left(u_{cc} - u\frac{\rho_{cc}}{\rho}\right)\left(u_{mm} - u\frac{\rho_{mm}}{\rho}\right) - \left(u_{cm} - u\frac{\rho_{cm}}{\rho}\right)^2 \geq 0, \tag{15}$$

and $c^* m^* \rho_c L_c/\theta - m^* L_k + f_k > 0$, then the trace is positive.

[7] Specifically (13) is written as:

$$f_{kk}(\rho_m\varphi_R - 1) + \rho_m\varphi_k + \rho_m f_k\varphi_c + f_k\rho_c > 0.$$

When $\rho_m\varphi_R < 1$, a larger $|f_{kk}|$ and a smaller $|\varphi_R|$, $|\varphi_c|$, and $|\varphi_k|$ make the BG condition attainable.

As for $c^*m^*\rho_c L_c/\theta - m^*L_k + f_k > 0$, we cannot give as intuitive a condition as the BG condition in the previous proposition. As a sufficient condition, we can say $c^*m^*\rho_c L_c/\theta - m^*L_k + f_k > 0$ if $\rho_c \geq 0$, $L_c \geq 0$, and $L_k \leq 0$. As discussed already, $\varphi_R = 1/L_m$, $\varphi_c = -L_c/L_m$ and $\varphi_k = -L_k/L_m$, and we have assumed that increasing the cost of holding money lowers money demand ($\varphi_R < 0$). Therefore $L_c \geq 0$ (resp. $L_k \leq 0$) indicates that money demand is nondecreasing (resp. nonincreasing) in consumption (resp. capital). Thus, we can obtain the following proposition:

Proposition 2. *Suppose that the assumptions in the previous subsection hold, that the condition (15) holds, and that money demand is nondecreasing in consumption and nonincreasing in capital. If* $\rho_c \geq 0$*, then* $\mathrm{tr}(J_{MIUF}) > 0$.

Equation (15) is established *either* when u and ρ are concave with respect to c and m in the case of $u < 0$ *or* when u is concave and ρ is convex in the case of $u > 0$. This is because a linear combination of concave functions ($u(c,m)$-$\rho(c,m)u(c^*,m^*)/\rho(c^*,m^*)$ for fixed c^* and m^*) is also a concave function. Thus, in the case of $u < 0$, the stronger is the concavity of the felicity function and the discounting function, the higher is the possibility of DMI in consumption. In the case of $u > 0$, the convexity of the discounting function matters instead.

Nondecreasing money demand in consumption $\varphi_c \geq 0$ may be a natural assumption as long as money and consumption are complements. The nominal rate is the marginal rate of substitution between consumption and real balances ($R = (u_c - \rho_c V(m+k))/(u_m - \rho_m V(m+k)) = L$). In general, increasing consumption raises the marginal utility of consumption and the marginal rate of substitution ($L_c > 0$), whereas increasing real balances raises the marginal utility of money but reduces the marginal rate of substitution ($L_m < 0$). Canceling out the effect of nominal interest rates leads to $\varphi_c = -L_c/L_m > 0$.

Because $\varphi_R = 1/L_m < 0$, $\lambda = u_c - \rho_c V(k+m) > 0$, and L_k is specified in equation (32) in Appendix A, nondecreasing money demand in capital is equivalent to:

$$\rho_c u_m - \rho_m u_c \;\; \leq \;\; 0, \tag{16}$$

for a steady state. It should be noted that the equality is established in a constant preference case ($\rho_c = \rho_m = 0$), in the Epstein–Hynes utility function case ($u = -1$), or in the Uzawa case ($\rho = \tilde{\rho}(u(c,m))$).

The condition (16) sheds new light on the comparative statics conducted by Chen et al. (2008). They claimed that a reverse Tobin effect emerges in the case of $\rho_m < 0$, independent of ρ_c, when they assumed the determinant of the Jacobian matrix to be negative. On the other hand, the condition (16) states that a negative ρ_m involves a negative ρ_c in the case of a positive marginal felicity. Furthermore, as claimed in Proposition 2, the steady state might be unstable in the case of $\rho_c < 0$. Unlike Chen et al., our paper claims that a combination of $\rho_m < 0$ and $\rho_c \geq 0$ is less plausible from a theoretical viewpoint.

We combine the two propositions to offer conditions for local stability. A sufficient condition for local stability is both a negative determinant and a positive trace of the Jacobian matrix. A sufficient condition for a negative determinant or the BG condition (13) is that $\rho_c \geq 0$, $\rho_m \geq 0$, $\varphi_c \geq 0$, and $\varphi_k \geq 0$. On the other hand, Proposition 2 requires $\rho_c \geq 0$, $\varphi_c \geq 0$ and $\varphi_k \leq 0$ for a positive trace of the Jacobian matrix. Thus, when considering the sufficient condition for local stability, the money demand function should be independent of capital ($\varphi_k = 0$) or $u_c/\rho_c = u_m/\rho_m$, indicating that the ratio of marginal felicity to marginal impatience with respect to consumption is equal to that with respect to real balances.

Summarizing the above discussion leads to the following theorem:

Theorem 3. *Suppose that the assumptions in the previous subsection are satisfied. 1) If the conditions (13) and (15) hold, the money demand function is nondecreasing in consumption, and* $\rho_c \geq 0$ *and* $\rho_c u_m \leq \rho_m u_c$ *for a steady state, then the steady state is locally stable. 2) When we also assume that* $\rho_m \geq 0$ *and* $u_c/\rho_c = u_m/\rho_m$*, the BG condition (13) is always assured.*

That is, as a sufficient condition, when condition (15) hold, $\varphi_c \geq 0$, $\rho_c \geq 0$, $\rho_m \geq 0$ and $u_c/\rho_c = u_m/\rho_m$, then the steady state leading to a Tobin effect is locally stable.

The assumption that $u_c/\rho_c = u_m/\rho_m$, leading to $L_k = 0$, makes the representation of L_m and L_c simpler, to give more specified sufficient conditions for local stability. As long as u_c, u_m, ρ_c, and ρ_m never takes a value of zero, then $L_m < 0$, $L_c \geq 0$ and (15) are represented as *either* $L_m = (u_c u_{mm} - u_m u_{cm})/u_c^2 < 0$ and $L_c = (u_c u_{cm} - u_m u_{cc})/u_c^2 \geq 0$ *or* $(\rho_c \rho_{mm} - \rho_m \rho_{cm})/\rho_c^2 < 0$ and $(\rho_c \rho_{cm} - \rho_m \rho_{cc})/\rho_c^2 \geq 0$. The former set of conditions corresponds to the case of Fischer (1979), whereas the latter corresponds to Epstein and Hynes (1983). A sufficient condition for the former is $u_{cm} \geq 0$ (Calvo, 1979), whereas that for the latter is $\rho_{cm} \geq 0$. Because we would like to give a detailed condition of ρ_c or ρ_m for local stability, we consider only the former representation. Consequently, under $u_c/\rho_c = u_m/\rho_m$, the decreasing money demand in nominal interest rates and the increasing money demand in consumption are equivalent to $u_c u_{mm} - u_m u_{cm} < 0$ and $u_c u_{cm} - u_m u_{cc} > 0$.

Using the representation, the BG condition (13) is satisfied when:

$$\begin{aligned} &\rho_c[f_{kk} u_c^2 u_m - f_k(2u_c u_m u_{cm} - u_m^2 u_{cc} - u_c^2 u_{mm}) \\ &\quad - f_{kk}(u_c^2 u_{mm} - u_c u_m u_{cm}) > 0, \end{aligned} \tag{17}$$

and the trace is positive if (as a sufficient condition):

$$\begin{aligned} &\frac{(u_{cc}/u - \rho_{cc}/\rho)(u_{mm}/u - \rho_{mm}/\rho) - (u_{cm}/u - \rho_{cm}/\rho)^2}{(u_c/u - \rho_c/\rho)^2} \\ &\quad + \frac{\rho_c}{u_c^2}(u_{cm} u_c - u_{cc} u_m) > 0 \end{aligned} \tag{18}$$

for a steady state. Both conditions are satisfied when $\rho_c \geq 0$ and *either* u and ρ are concave in the case of $u < 0$ *or* u is concave and ρ is convex in the case of $u > 0$. In the condition (18), it remains difficult to provide an explicit expression of ρ_c. However, it is much easier to check numerically whether there exists a negative ρ_c such that (17) and (18) than is the case when $\rho_c u_m \neq \rho_m u_c$.

Summarizing the above discussion leads to the following theorem:

Theorem 4. *Suppose that the first two assumptions in the previous subsection are satisfied. 1) If $\rho_c u_m = \rho_m u_c$ and the conditions (17) and (18) hold for a steady state, then the steady state is locally stable. 2) Further assume that $\rho_c (= \rho_m u_c / u_m) \geq 0$, u is concave,* either *$u < 0$ and ρ is concave* or *$u > 0$ and ρ is convex, and ,* either *$u_c u_{mm} - u_m u_{cm} < 0$ and $u_c u_{cm} - u_m u_{cc} > 0$* or *$u_{cm} > 0$ for a steady state. Then the conditions (17) and (18) are no longer needed for local stability.*

This is consistent with Uzawa (1968) in the case of $u > 0$, $\rho_c > 0$, $\rho_{cc} > 0$, and $\rho_c u_m = \rho_m u_c$, and with Fischer (1979) and Calvo (1979) in the case of a constant time preference ($\rho_c = \rho_m = 0$).

We can easily derive an impatience representation of the theorem as a corollary.

Corollary 5. *Suppose that the first two assumptions in the previous subsection are satisfied. If $\rho_c \geq 0$, $\rho_c u_m = \rho_m u_c$, u is concave,* either *$u < 0$ and ρ is concave* or *$u > 0$ and ρ is convex, and* either *$\rho_c \rho_{mm} - \rho_m \rho_{cm} < 0$ and $\rho_c \rho_{cm} - \rho_m \rho_{cc} > 0$* or *$\rho_{cm} > 0$ for a steady state, then the steady state is locally stable.*

This is consistent with Epstein and Hynes (1983) assuming $u = -1$. It should be noted that ρ_c or ρ_{cc} should be selected so that $u_c - u\rho_c/\rho > 0$ and $u_{cc} - u\rho_{cc}/\rho < 0$ in either case.

Before closing this section, we mention a sufficient condition for the concavity of the Hamiltonian (3). As discussed before, when this concavity condition is imposed, the case with $u > 0$ does not survive. In addition, we should add the assumption that $u(u_{cc}u_{mm} - u_{cm}^2) + u_c(u_m u_{cm} - u_{mm}u_c) + u_m(u_c u_{cm} - u_{cc}u_m) > 0$ to theorem 4. One example of functional forms satisfying the additional assumption as well as the assumptions in the theorem 4 is that $u(c, m) = (c^{1-\alpha}m^{\alpha})^{1-\sigma}/(1-\sigma)$ and $\rho = \tilde{\rho}(c^{1-\alpha}m^{\alpha})$ for $\sigma > 1$ and $0 < \alpha < 1$. It should be noted that this felicity satisfies $u_c/\rho_c = u_m/\rho_m$.

3 The TC model

This section deals with the TC model, in which purchasing consumption goods involves transactions costs. Money is demanded because it reduces these costs. Our paper follows the model structures and most notations in Chen et al. (2008). This section follows the same order of discussion developed in the previous section. The first subsection describes the model economy and then defines a steady state, and the second subsection examines the local dynamic properties. Some explanations are omitted to avoid repetition.

3.1 The model economy

As in the MIUF model, the economy has a constant population and no technological progress, and involves homogeneous consumers and the government. The representative agent with perfect foresight solves the following problem:

$$\max_{\{c_t, m_t\}} \int_0^\infty e^{-\Delta_t} u(c_t) dt, \quad \text{s.t.}$$

$$\dot{a}_t = f(k_t) - \pi_t m_t + v_t - c_t - T(c_t, m_t), \tag{19}$$

$$\dot{\Delta}_t = \rho(c_t), \tag{20}$$

where $k_0 > 0$ and $m_0 > 0$ are given, T is the transaction function, and the other notations c, m, k, a, π, v, u, ρ, f, and Δ are the same as in the previous section. We assume that u, ρ, f, and T are twice continuously differentiable. We assume that $u \neq 0$, $u_c \geq 0$, $u_{cc} \leq 0$, and $\rho > 0$ for all $c > 0$, and $f > 0$ for all $k > 0$. The conditions of u, ρ, and f are the same as in the MIUF model except that u and ρ are a function only of consumption. We assume that $T > 0$, $T_c > 0$, $T_m < 0$ for $c, m > 0$. There exist positive transaction costs, T, for a positive consumption. Such costs are raised by consumption but reduced by real balances. Other assumptions regarding u, ρ, f, and T are imposed later.

Let the present-value Hamiltonian be:

$$H = e^{-\Delta}\{u(c) + \lambda(f(k) - \pi m + v - c - T(c, m)) - \phi\rho(c) + \psi(a - m - k)\}, \tag{21}$$

where ψ is the multiplier, and λ and ϕ denote the costate variable associated with (19) and (20). Optimal behavior by the consumer implies the set of necessary optimality conditions:

$$u_c - \lambda(1 + T_c) - \phi\rho_c = 0 \tag{22}$$

$$\lambda(f_k + \pi + T_m) = 0 \tag{23}$$

(6), (7), (19), (20), $k_0 > 0$, $m_0 > 0$, $\lim \lambda_t a_t e^{-\Delta_t} = 0$ and $\lim \phi_t \Delta_t e^{-\Delta_t} = 0$. As in the previous section, ϕ_t, the lifetime utility from the period t, is equal to $V(a_t)$ and λ_t is equal to $V'(a_t)$.

The government acts in the same way as described in the previous section. In equilibrium, the money market condition is unaltered (9) but the goods market condition is:

$$\dot{k} = f(k) - c - T(c, m). \tag{24}$$

A steady state is defined in the same way as in the previous section, and we also assume that there exists a steady state such that: $c^* > 0$, $m^* > 0$, $k^* > 0$, $\lambda^* = (u_c - \phi^*\rho_c)/(1 + T_c) > 0$, $\phi^* = u^*/\rho^*$, $f_k > 0$, and $\pi^* + f_k > 0$. As $T_c > 0$, we need $u_c - \rho_c u/\rho > 0$ for $\lambda^* > 0$. This is satisfied when $u < 0$ and $\rho_c > 0$.

We assume that $u_{cc} - \rho_{cc} u/\rho < 0$ and $T_{cc} > 0$ for a steady state. Those are sufficient conditions for the positiveness of the intertemporal elasticity of substitution. The former condition is established either if $\rho_{cc} > 0$ and $u > 0$ or if $\rho_{cc} < 0$ and $u < 0$.

As in the previous section, we define the money demand function φ from (23) as $R_t + T_m(c, \varphi(c, R)) = 0$ implicitly. Unlike in the MIUF model, the money demand function is independent of capital, and characterized not by preferences (u and ρ), but only by the transaction cost technology T. The total differential is $dR + T_{mm}dm + T_{cm}dc = 0$. The implicit function theorem implies that $\varphi_R = -1/T_{mm}$. We assume that $0 < T_{mm} < \infty$ for $-\infty < \varphi_R < 0$. We can also demonstrate that $\varphi_c = -T_{cm}/T_{mm}$. The sign of T_{cm} determines whether money demand is increasing in consumption, but we provide a restriction on T_{cm} in the next subsection.

We summarize the assumptions on u, ρ, f, T, and φ that we have made so far.

1. $u \neq 0$, $u_c \geq 0$, $u_{cc} \leq 0$, and $\rho > 0$ for all $c > 0$, and $f > 0$ for all $k > 0$.

2. $f_k > 0$, $\mu + f_k > 0$, $f_{kk} < 0$, $u_c - u\rho_c/\rho > 0$, and $u_{cc} - u\rho_{cc}/\rho < 0$ for a steady state.

3. $T > 0$, $T_c > 0$, $T_m < 0$ for all c and $m > 0$, and $T_{cc} > 0$ and $0 < T_{mm} < \infty$ or equivalently $-\infty < \varphi_R < 0$ for a steady state.

3.2 Local stability conditions

In this subsection, we examine local stability conditions, as in the previous section. We provide a linear approximation of the dynamic system around the steady state using the Jacobian matrix, and reproduce the comparative statics of Chen et al. (2008). Then, we present the condition for a negative determinant and a positive trace of the matrix as two propositions. We also provide a theorem combining the two propositions, and a corollary.

Appendix B gives a linearized dynamic system around the steady state:

$$\begin{bmatrix} \dot{c} \\ \dot{m} \\ \dot{k} \end{bmatrix} = J_{TC} \begin{bmatrix} c - c^* \\ m - m^* \\ k - k^* \end{bmatrix},$$

where J_{TC} is specified in (39) of Appendix B. In addition, Appendix B conducts a comparative static analysis with respect to the rate of money growth:

$$\begin{bmatrix} dc^*/d\mu \\ dm^*/d\mu \\ dk^*/d\mu \end{bmatrix} = \frac{c^* m^*}{\theta \det(J_{TC})} \begin{bmatrix} T_m f_k \\ \rho_c f_k - (1 + T_c) f_{kk} \\ -\rho_c f_k \end{bmatrix}, \tag{25}$$

where $\theta = \tilde{\theta} + cT_{cc}/(1 + T_c)$ and $\tilde{\theta} = -c(u_{cc} - \rho_{cc}V)/(u_c - \rho_c V)$ represents the intertemporal elasticity of substitution. Note that $\theta > 0$ by assumption. Assuming that $\det(J_{TC}) < 0$, the monetary expansion effect is the same as in Table 1 in Chen et al. (2008). The TC model also predicts a positive (resp. negative) correlation when impatience increases (resp. decreases) with consumption.

Let us examine the dynamic system. Like the MIUF model, the TC model requires only one negative root in the three-dimensional system from the viewpoint of economic theory. As sufficient conditions, the determinant

$\det(J_{TC})$ should be negative and the trace $\text{tr}(J_{TC})$ should be positive. Similarly to Section 2.2, Section 3.2 examines conditions for $\det(J_{TC}) < 0$ and $\text{tr}(J_{TC}) > 0$ separately, and then combines them to provide the main result in this section as a theorem along with a corollary.

First, we present conditions for $\det(J_{TC}) < 0$. With some algebra (see Appendix B):

$$\det(J_{TC}) = \frac{c^* m^*}{\theta}\{f_{kk}(T_{mm}(1+T_c) - T_{cm}T_m) - \rho_c(T_{mm}f_k + T_m f_{kk})\}. \quad (26)$$

Note that $\theta > 0$ by assumption. Thus, if:

$$\rho_c > \frac{f_{kk}\{(1+T_c) - T_{cm}T_m/T_{mm}\}}{f_k + f_{kk}T_m/T_{mm}}, \quad (27)$$

then $\det(J_{TC}) < 0$.

As long as $(1+T_c) - T_{cm}T_m/T_{mm} > 0$, then $\rho_c < 0$ and $\det(J_{TC}) < 0$ for a suitable choice of a function ρ. This condition is required in the case of the TC model with a constant time preference ($\rho_c = 0$). To interpret $(1+T_c) - T_{cm}T_m/T_{mm} > 0$, we use the money demand function φ. As φ is defined as $R_t + T_m(c, \varphi(c, R)) = 0$ and $\varphi_c = -T_{cm}/T_{mm}$, the condition is equivalent to $1+T_c+T_m\varphi_c = d(c+T)/dc > 0$. The differentiation indicates that if $T_{cm} \geq 0$, a direct increase in consumption raises transaction costs, but an indirect increase in money via the money demand function reduces transaction costs. Thus, the above condition is satisfied *either* if money demand decreases with consumption *or* if the marginal increase in consumption is not very large.

When $c + T$ is called "gross" consumption, we can say that an increase in "net" consumption strictly raises "gross" consumption for a given nominal rate of interest. This is more likely to hold for a smaller $|T_{cm}|$ and a larger $|T_{mm}|$, implying that the money demand function is less elastic with respect to consumption and nominal interest rates.

As in the MIUF model, (27) is a monetary version of the BG condition. To see this, assume that $(1 + T_c) - T_{cm}T_m/T_{mm} > 0$. Then, (27) can be rewritten as:

$$f_{kk} < \rho_c \frac{f_k + f_{kk}T_m\varphi_R}{1 + T_c + T_m\varphi_c} = \rho_c \frac{d(c+T)/dk}{d(c+T)/dc} = \rho_c \frac{dc}{dk}.$$

Note that $R = \mu + f_k$ and $c + T = f(k)$ in a steady state. The effect of capital on "net" consumption is described via "gross" consumption. The condition (27) means the increase in the discount rate dominates the increase in the marginal product of capital for a steady state.

We can summarize the above discussion as the following proposition:

Proposition 6. *Suppose that the assumptions in the previous subsection hold. If the BG condition (27) holds, then* $\det(J_{TC}) < 0$.

Like the MIUF model, the TC model requires the dominance of the discount rate over the marginal product of capital for $\det(J_{TC}) < 0$. The right-hand side of (27) is described only by the technology functions f and T, and is independent of the preference function u or ρ. A negative value of ρ_c, leading to a reverse Tobin effect, is possible as long as an increase in "net" consumption strictly raises "gross" consumption for a given nominal rate of interest ($(1+T_c) - T_{cm}T_m/T_{mm} > 0$), and is more plausible as the curvature of the production function is higher and the money demand function is less elastic with respect to consumption and nominal interest rates. Even though a comparative static analysis is justified, we need to investigate $\mathrm{tr}(J_{TC}) > 0$ for local stability.

We examine conditions for $\mathrm{tr}(J_{TC}) > 0$. Appendix B demonstrates that:

$$\begin{aligned}\mathrm{tr}(J_{TC}) = {} & \frac{(1+T_c)(u_{cc} - \rho_{cc}u/\rho)T_{mm}}{(1+T_c)(u_{cc} - \rho_{cc}u/\rho) - (u_c - \rho_c u/\rho)T_{cc}} m^* \\ & - \frac{(u_c - \rho_c u/\rho)(T_{cc}T_{mm} - T_{cm}^2 - \rho_c T_{cm})}{(1+T_c)(u_{cc} - \rho_{cc}u/\rho) - (u_c - \rho_c u/\rho)T_{cc}} m^* + f_k, \end{aligned} \tag{28}$$

for a steady state. We have already assumed that $T_c > 0$, $T_{cc} > 0$, $T_{mm} > 0$, $f_k > 0$, $u_c - u\rho_c/\rho > 0$, and $u_{cc} - u\rho_{cc}/\rho < 0$ for a steady state in the previous subsection. Thus, if $T_{cc}T_{mm} - T_{cm}^2 - \rho_c T_{cm} \geq 0$, the trace is positive. As discussed, $\varphi_c = -T_{cm}/T_{mm}$ from the implicit function theorem. Because $T_{cm} \leq 0$ means that money makes consumption easier, increasing money demand in consumption would be as plausible as in the MIUF model.

Thus, we obtain the following proposition:

Proposition 7. *Assume the assumptions in the previous subsection hold. If the money demand function is nondecreasing in consumption ($T_{cm} \leq 0$), and*

$$\rho_c \geq \frac{T_{cc}T_{mm} - T_{cm}^2}{T_{cm}},$$

then $\mathrm{tr}(J_{TC}) > 0$*.*

In order for ρ_c to take a negative value at the steady state, $T_{cc}T_{mm} - T_{cm}^2 > 0$, which holds when T is convex.[8] As long as the transaction cost function is strongly convex and the money demand function is less elastic with respect to consumption, then there is potential for DMI in consumption.

As discussed in Propositions 6 and 7, we require that $(1+T_c) - T_{cm}T_m/T_{mm} > 0$ and $T_{cc}T_{mm} - T_{cm}^2 > 0$ for ρ_c containing zero. Combining Propositions 6 and 7 gives the following theorem:

Theorem 8. *Suppose that the assumptions in the previous subsection are satisfied. 1) If the money demand function is nondecreasing in consumption* $(T_{cm} \leq 0)$*, and*

$$\rho_c \geq \max\left[\frac{f_{kk}\{(1+T_c) - T_{cm}T_m/T_{mm}\}}{f_k + f_{kk}T_m/T_{mm}}, \frac{T_{cc}T_{mm} - T_{cm}^2}{T_{cm}}\right]$$

for a steady state, then the steady state is locally stable. 2) When an increase in "net" consumption is additionally assumed to raise "gross" consumption for given nominal interest rates $(T_{mm}(1+T_c) - T_{cm}T_m > 0)$*, and* T *is convex* $(T_{cc}T_{mm} - T_{cm}^2 > 0)$*, then* ρ_c *can take a negative value.*

Unlike in the case of the MIUF model, we can determine the range of ρ_c explicitly using only the production function f and the transaction costs technology T. Once the steady state is obtained, it is much easier than in the

[8] We have already assumed that $T_{cc} > 0$ and $T_{mm} > 0$.

MIUF model to check whether the comparative statics with DMI is justified and whether the steady state is stable.

As shown, the BG condition with DMI is more likely to hold as the production function becomes more concave and as the money demand function becomes less elastic, or if there is a small $|T_{cm}|$ and a large $|T_{mm}|$. As $|T_{cm}|$ becomes smaller and $|T_{mm}|$ becomes larger, the transaction function becomes more convex, making a positive trace of the Jacobian matrix more plausible. When holding money is relatively costless and ineffectual in making consumption easier, the local stability under DMI is achievable.

As a simpler condition for local stability, we offer the following corollary:

Corollary 9. *Suppose that the assumptions in the previous subsection hold, and T is convex ($T_{cc}T_{mm} - T_{cm}^2 \geq 0$), $T_{cm} \leq 0$, and $T_{mm}(1+T_c) - T_{cm}T_m > 0$. If $\rho_c \geq 0$, then a steady state is locally stable.*

It should be noted that ρ_c or ρ_{cc} should be selected so that $u_c - \rho_c u/\rho > 0$ and $u_{cc} - \rho_{cc}u/\rho < 0$ in either case. The two conditions always hold when $u < 0$, $\rho_c > 0$, and $\rho_{cc} < 0$.

Before closing this section, we mention a sufficient condition for the concavity of the Hamiltonian (21). The condition is that $u < 0$, $uu_{cc} - u_c^2 > 0$, ρ is concave, and T is convex. Thus, if the sufficiency condition is imposed, the case with $u > 0$ does not survive. In the negative felicity case, we should add the assumption that $uu_{cc} - u_c^2 > 0$ to Theorem 8. One example of functional forms satisfying all the conditions is $u(c) = c^{1-\sigma}/(1-\sigma)$, where $\sigma > 1$, and

$T(c, m) = T_0(m/c)^{-\eta}c$, where $T_0 > 0$ and $\eta > 0$. This convex transaction cost function always satisfies $T_{mm}(1 + T_c) - T_{cm}T_m > 0$.

4 Concluding remarks

We have examined the local stability properties in the MIUF and TC models with recursive utility, utility with an internally determined discount rate. A sufficient condition for local stability in an economic theoretical sense is that the determinant of the Jacobian matrix is negative and the trace is positive. Assuming the existence of the steady state, we have found that a decreasing money demand function in the nominal interest rates and a monetary version of the BG condition, in which the increase in the discount rate dominates the increase in the marginal product in the steady-state equilibrium, ensure a negative determinant of the Jacobian matrix of the dynamic system, and that such conditions justify the comparative statics by Chen et al. (2008).

In the MIUF model, a sufficient condition for local stability is that impatience is marginally increasing in consumption and real balances, the money demand function is nondecreasing in consumption and independent of capital, the felicity function is concave with respect to consumption and real balances, and the discount rate function is concave (resp. convex) in the case of negative (resp. positive) felicity. The established conditions are consistent with Uzawa (1968), Fischer (1979), Calvo (1979) and Epstein and Hynes (1983).

In the TC model, a sufficient condition for local stability is IMI in consumption, and nondecreasing money demand in consumption. When an increase in "net" consumption raises "gross" consumption for given nominal interest rates, and the transaction cost function is convex, then the case of a constant time preference is also stable.

Although marginal impatience should be increasing in consumption and money as a sufficient condition for local stability, we do not deny the possibility of a negative case. Our analysis has demonstrated that DMI in consumption and real balances is more plausible as: the concavity of the production function becomes stronger and the money demand function becomes less elastic in both models; the curvatures of the felicity and the discount rate functions become stronger in the MIUF model; the concavity of the transaction costs function becomes stronger in the TC model.

Our analysis has suggested a possibility for DMI. Accordingly, the dynamic systems of both models might be locally stable in the case of DMI in consumption and real balances, which receives more support in terms of empirical evidence. Conducting several quantitative exercises with a set of parameters reconciling empirical results (Becker and Mulligan, 1997; Ogaki and Atkinson, 1997; and some authors) may be an important future task.

Appendix

In this Appendix, we provide some indicative mathematical manipulations. Section A deals with the MIUF model, and Section B with the TC model.

A. The MIUF model

As discussed in subsection 2.1, the dynamic system in equilibrium is described by (4), (5), (6), (9), (10), $\phi = V(m+k)$, $\lambda = V'(k+m) = u_c - \rho_c V$, and the boundary conditions ($k_0 > 0$, $m_0 > 0$, $\lim_{t\to\infty} \lambda(k+m)e^{-\Delta} = 0$ and $\lim_{t\to\infty} \phi\Delta e^{-\Delta} = 0$, where $\Delta_t = -\int_0^t \rho(c_\tau, m_\tau)d\tau$).

Combining (4) and (5) by replacing for λ yields $\pi = (u_m - \rho_m V)/(u_c - \rho_c V) - f_k$. Substituting this equation into (9) gives:

$$\dot{m}/m = \mu + f_k - \frac{u_m - \rho_m V}{u_c - \rho_c V}. \tag{29}$$

Taking the time derivatives in (4), and using $\lambda = V'(k+m) = u_c - \rho_c V$ and (6), we obtain:

$$(u_{cc} - \rho_{cc}V)\dot{c} + (u_{cm} - \rho_{cm}V)\dot{m} - \rho_c(u_c - \rho_c V)\dot{k} - \rho_c(u_c - \rho_c V)\dot{m} = (u_c - \rho_c V)(\rho - f_k).$$

In sum, equilibrium c, m, k solves the first-order differential equations:

$$\theta\dot{c}/c = f_k - \rho + \frac{u_{cm} - \rho_{cm}V}{u_c - \rho_c V}\dot{m} - \rho_c\dot{k} - \rho_c\dot{m}, \tag{30}$$

(10) and (29) subject to the boundary conditions, where $\theta = -(u_{cc} - \rho_{cc}V)c/(u_c - \rho_c V)$ represents the intertemporal elasticity of substitution.

Linear approximation

The linear approximation around the steady state c^*, m^*, k^* gives:

$$\begin{bmatrix} \theta/c^* & \rho_c - K(c^*, m^*) & \rho_c \\ 0 & 1/m^* & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{c} \\ \dot{m} \\ \dot{k} \end{bmatrix} = \begin{bmatrix} -\rho_c & -\rho_m & f_{kk} \\ -L_c & -L_m & -L_k + f_{kk} \\ -1 & 0 & f_k \end{bmatrix} \begin{bmatrix} c_t - c^* \\ m_t - m^* \\ k_t - k^* \end{bmatrix}, \tag{31}$$

where $K = (u_{cm} - \rho_{cm}V)/(u_c - \rho_c V)$, and $L_x = \partial\{(u_m - \rho_m V)/(u_c - \rho_c V)\}/\partial x$ for $x = k, c, m$, represented as follows:

$$L_k = \frac{\rho_c u_m - \rho_m u_c}{u_c - \rho_c V}, \tag{32}$$

$$L_c = \frac{(u_c - \rho_c V)(u_{cm} - V\rho_{cm}) - (u_m - \rho_m V)(u_{cc} - V\rho_{cc})}{(u_c - \rho_c V)^2}, \tag{33}$$

$$L_m = \frac{(u_c - \rho_c V)(u_{mm} - V\rho_{mm}) - (u_m - \rho_m V)(u_{cm} - V\rho_{cm})}{(u_c - \rho_c V)^2} + L_k. \tag{34}$$

We denote the 3×3 matrix on the left-hand side of (31) by A_{MIUF} and the 3×3 matrix on the right-hand side of (31) by B_{MIUF}. Then,

$$\begin{bmatrix} \dot{c} \\ \dot{m} \\ \dot{k} \end{bmatrix} = J_{MIUF} \begin{bmatrix} c_t - c^* \\ m_t - m^* \\ k_t - k^* \end{bmatrix},$$

where J_{MIUF} is given by:

$$J_{MIUF} = A_{MIUF}^{-1} B_{MIUF}. \tag{35}$$

The determinant

The determinant is represented by:

$$\det(J_{MIUF}) = \frac{\det(B_{MIUF})}{\det(A_{MIUF})} = \frac{1}{\det(A_{MIUF})} \det \begin{bmatrix} -\rho_c & -\rho_m & f_{kk} \\ -L_c & -L_m & -L_k + f_{kk} \\ -1 & 0 & f_k \end{bmatrix}.$$

With some algebra, we can obtain (12) in Section 2.2.

The trace

Because:

$$A^{-1} = \begin{bmatrix} \frac{c^*}{\theta} & -\frac{c^* m^* \rho_c}{\theta} + \frac{c^* m^*}{\theta} K & -\frac{c^* \rho_c}{\theta} \\ 0 & m^* & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

the trace is:

$$\begin{aligned} \mathrm{tr}(J_{MIUF}) &= \mathrm{tr}(A_{MIUF}^{-1} B_{MIUF}) \\ &= \frac{c^* m^*}{\theta} \rho_c L_c - \frac{c^* m^*}{\theta} K(c^*, m^*) L_c - m^* L_m + f_k \\ &= \frac{c^* m^*}{\theta} \rho_c L_c - m^* L_k + f_k \\ &\quad + m^* \frac{(u_{cm} - \rho_{cm} V)^2 - (u_{cc} - \rho_{cc} V)(u_{mm} - \rho_{mm} V)}{(u_c - \rho_c V)(u_{cc} - \rho_{cc} V)} \end{aligned}$$

Using $V = u/\rho$ at the steady state, we can obtain (14) in Section 2.2.

Comparative statics

Under a steady state, in which $\dot{c} = \dot{m} = \dot{k} = 0$, the total differentiation of (30), (29), and (10) with respect to c, m, k, and μ yields:

$$B_{MIUF} \begin{bmatrix} dc \\ dm \\ dk \end{bmatrix} + \begin{bmatrix} 0 \\ d\mu \\ 0 \end{bmatrix} = 0.$$

Therefore,

$$\begin{bmatrix} dc/d\mu \\ dm/d\mu \\ dk/d\mu \end{bmatrix} = B_{MIUF}^{-1} \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} = \frac{1}{\det(B_{MIUF})} \begin{bmatrix} -\rho_m f_k \\ -(f_{kk} - \rho_c f_k) \\ -\rho_m \end{bmatrix}$$

Thus, we can obtain (11) in Section 2.2 because $\det(J_{MIUF}) = \det(B_{MIUF})/\det(A_{MIUF})$ and $\det(A_{MIUF}) = \theta/(c^* m^*)$.

B. The TC model

As discussed in subsection 3.1, the dynamic system in equilibrium is described by (6), (9), (22), (23), (24), $\phi = V(m+k)$, $\lambda = V'(k+m) = u_c - \rho_c V$, and the boundary conditions ($k_0 > 0$, $m_0 > 0$, $\lim_{t\to\infty} \lambda(m+k)e^{-\Delta} = 0$ and $\lim_{t\to\infty} \phi \Delta e^{-\Delta} = 0$, where $\Delta_t = -\int_0^t \rho(c_\tau, m_\tau) d\tau$).

Combining (22) and (23) by replacing for λ leads to:

$$\dot{m}/m = \mu + f_k + T_m. \tag{36}$$

Taking the time derivatives in (22), and using $\lambda = (u_c - \rho_c V)/(1+T_c) = V'(m+k)$ and (6), we obtain:

$$\frac{(u_{cc} - \rho_{cc} V)\dot{c}}{1+T_c} - \frac{u_c - \rho_c V}{(1+T_c)^2}\{T_{cc}\dot{c} + T_{cm}\dot{m} + \rho_c(\dot{k} + \dot{m})\} = \frac{u_c - \rho_c V}{1+T_c}(\rho - f_k).$$

In sum, equilibrium c, m, k solves the first-order differential equations:

$$\theta \dot{c}/c = f_k - \rho - \frac{T_{cm}}{1+T_c}\dot{m} - \frac{\rho_c}{1+T_c}\dot{m} - \frac{\rho_c}{1+T_c}\dot{k}, \tag{37}$$

(24) and (36) subject to the boundary conditions, where $\theta = \tilde{\theta} + cT_{cc}/(1+T_c)$ and $\tilde{\theta} = -c(u_{cc} - \rho_{cc}V)/(u_c - \rho_c V)$ represents the intertemporal elasticity of substitution.

Linear approximation

The linear approximations around the steady state c^*, m^*, k^* gives:

$$\begin{bmatrix} \theta/c^* & \frac{\rho_c + T_{cm}}{1+T_c} & \frac{\rho_c}{1+T_c} \\ 0 & 1/m^* & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{c} \\ \dot{m} \\ \dot{k} \end{bmatrix} = \begin{bmatrix} -\rho_c & 0 & f_{kk} \\ T_{cm} & T_{mm} & f_{kk} \\ -(1+T_c) & -T_m & f_k \end{bmatrix} \begin{bmatrix} c_t - c^* \\ m_t - m^* \\ k_t - k^* \end{bmatrix}. \tag{38}$$

We denote the 3×3 matrix on the left-hand side of (38) by A_{TC} and the 3×3 matrix on the right-hand side of (38) by B_{TC}. That is:

$$\begin{bmatrix} \dot{c} \\ \dot{m} \\ \dot{k} \end{bmatrix} = J_{TC} \begin{bmatrix} c_t - c^* \\ m_t - m^* \\ k_t - k^* \end{bmatrix},$$

where J_{TC} is given by:

$$J_{TC} = A_{TC}^{-1} B_{TC}. \tag{39}$$

The determinant

The determinant is represented by:

$$\det(J_{TC}) = \frac{\det(B_{TC})}{\det(A_{TC})} = \frac{1}{\det(A_{TC})} \det \begin{bmatrix} -\rho_c & 0 & f_{kk} \\ T_{cm} & T_{mm} & f_{kk} \\ -(1+T_c) & -T_m & f_k \end{bmatrix}.$$

With some algebra, we can obtain (26) in Section 3.2.

The Trace

Because:

$$A_{TC}^{-1} = \begin{bmatrix} \frac{c^*}{\theta} & -\frac{c^* m^* (T_{cm}+\rho_c)}{\theta(1+T_c)} & -\frac{c^*}{\theta(1+T_c)}\rho_c \\ 0 & m^* & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

the trace is:

$$\begin{aligned} \mathrm{tr}(J_{TC}) &= \mathrm{tr}(A_{TC}^{-1} B_{TC}) = -\frac{c^* m^* (T_{cm}+\rho_c) T_{cm}}{\theta(1+T_c)} + m^* T_{mm} + f_k \\ &= \frac{(1+T_c)(u_{cc} - V\rho_{cc}) T_{mm}}{(1+T_c)(u_{cc} - V\rho_{cc}) - (u_c - V\rho_c) T_{cc}} m^* \\ &\quad - \frac{(u_c - V\rho_c)(T_{cc} T_{mm} - T_{cm}^2 - \rho_c T_{cm})}{(1+T_c)(u_{cc} - V\rho_{cc}) - (u_c - V\rho_c) T_{cc}} m^* + f_k. \end{aligned}$$

Using $V = u/\rho$ at the steady state, we can obtain (28) in Section 3.2.

Comparative statics

Under a steady state, in which $\dot{c} = \dot{m} = \dot{k} = 0$, the total differentiation of (37), (36), and (24) with respect to c, m, k, and μ yields:

$$B_{TC} \begin{bmatrix} dc \\ dm \\ dk \end{bmatrix} + \begin{bmatrix} 0 \\ d\mu \\ 0 \end{bmatrix} = 0.$$

Therefore:

$$\begin{bmatrix} dc^*/d\mu \\ dm^*/d\mu \\ dk^*/d\mu \end{bmatrix} = B_{TC}^{-1} \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} = \frac{1}{\det(B_{TC})} \begin{bmatrix} T_m f_k \\ \rho_c f_k - (1+T_c) f_{kk} \\ -\rho_c T_m \end{bmatrix}$$

Thus, we can obtain (25) in Section 3.2 because $\det(J_{TC}) = \det(B_{TC})/\det(A_{TC})$ and $\det(A_{TC}) = \theta/(c^* m^*)$.

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