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1 Introduction

Natural deduction and sequent calculus are logical systems which were both introduced by Gentzen [2]. He proved Hauptsatz, or cut elimination theorem, in sequent calculus for classical and intuitionistic logic. After Gentzen's foundation, Prawitz [4] proved normalization theorem, which is the representation of Gentzen's Hauptsatz in the system of natural deduction. Moreover, Prawitz has attempted to extend normalization theorem to modal logic including the system S4. He defined three formalization of S4 in natural deduction, and stated that the first and second ones did not enjoy normalization theorem but the last one did.

Recently, Medeiros [3] pointed out an error in the Prawitz's proof of normalization theorem for modal logic S4, and proved the theorem with a newly defined formalization of the same logic S4. However, the proof of Medeiros does not work too, as we will see in this note. Problems arise with classical absurdity rule. We may recognize this rule as structural rule in a sense, with the expansion of the definition of Prawitz's *segment* ([1]).

In the following, first we sketch the new formalization of S4 and its normalization procedure introduced by Medeiros. Second we indicate some cases which make troubles in the induction proof in [3].

2 New formalization of S4 by Medeiros

In this section, we present the formalization by Medeiros [3], called **NS4**, which is a logical system in natural deduction for classical propositional modal logic S4.

NS4 has $\wedge, \vee, \rightarrow, \bot, \Box$ as logical symbols, and the inference rules for introduction and elimination of $\wedge, \vee, \rightarrow$ in the system are defined as usual. The rules for introduction and elimination of the modal operator $\Box$ are defined as below.

- Introduction rule for $\Box$ is:

$$\dfrac{\Box B_1 \ldots \Box B_n \qquad \begin{matrix}[\Box B_1] \ldots [\Box B_n] \\ A\end{matrix}}{\Box A}\ (\Box I)\quad ,$$

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where A depends on no assumptions other than $\Box B_1, \ldots \Box B_n$, and these assumptions are all discharged at the rule $(\Box I)$.

- Elimination rule for $\Box$ is:

$$\frac{\Box A}{A}\ (\Box E).$$

Further, the system has the inference rule so called classical absurdity rule:

$$\begin{array}{c}[\neg A]\\ \dfrac{\bot}{A}\ (\bot_c)\end{array}.$$

Here we have to recall that the formula $(A \to \bot)$ may be abbreviated to $\neg A$.

The normalization procedure for a derivation Π is defined according to the form of maximal formula. We indicate only few cases which cause problematic situations.

2.1

In the case that Π is a critical derivation, and F is one of the major premiss of the last inference r of Π, whose degree is equal to that of Π, moreover, F is the conclusion of a $(\bot_c)$, say i, and the side-set formula of the assumption discharged at i is not the conclusion of an introduction rule:

$$\Pi \equiv \dfrac{\dfrac{\begin{array}{c}\Sigma_{0,1}\\ \dfrac{F \quad [\neg F]^i}{\bot}\\ \Sigma_{0,2}\\ \bot\end{array}}{F}\ i \qquad \begin{array}{c}\Sigma_1 \qquad \Sigma_n\\ H_1 \ldots H_n\end{array}}{C}\ r$$

is reduced to

$$\Pi' \equiv \begin{array}{c}\dfrac{\begin{array}{c}\Sigma_{0,1}\,\Sigma_1 \qquad \Sigma_n\\ F\ \ H_1 \ldots H_n\end{array}}{\bot}\ r\\ \Sigma_{0,2}\\ \bot\end{array} \qquad \text{or} \qquad \Pi' \equiv \dfrac{\begin{array}{c}\dfrac{\dfrac{\begin{array}{c}\Sigma_{0,1}\,\Sigma_1 \qquad \Sigma_n\\ F\ \ H_1 \ldots H_n\end{array}}{C}\ r \qquad [\neg C]^i}{\bot}\\ \Sigma_{0,2}\\ \bot\end{array}}{C}\ i$$

2.2

In the case that Π is a critical derivation whose last inference is an introduction rule of $\Box$, say r, and $\Box A$ is one of the major premiss of r whose degree is equal to that of Π, moreover, $\Box A$ is the conclusion of a

($\bot_c$), say i, and the side-set formula of the assumption discharged at i is the conclusion of an introduction rule of $\Box$:

$$\Pi \equiv \dfrac{\dfrac{\begin{array}{c}\dfrac{\dfrac{\begin{array}{c}\Lambda_1\\ \Box G_1\end{array}\ \ldots\ \begin{array}{c}\Lambda_m\\ \Box G_m\end{array}\quad \begin{array}{c}[\Box G_1]^k\ldots[\Box G_m]^k\\ \Lambda_{m+1}\\ A\end{array}}{\Box A}\,k \qquad [\neg\Box A]^i}{\bot}\\ \Sigma_0\\ \bot\end{array}}{\Box A}\,i \qquad \begin{array}{c}\Sigma_1\\ \Box B_1\end{array}\ldots\begin{array}{c}\Sigma_n\\ \Box B_n\end{array} \qquad \begin{array}{c}[\Box A]^r[\Box B_1]^r\ldots[\Box B_n]^r\\ \Sigma_{n+1}\\ B\end{array}}{\Box B}\,r$$

is reduced to

$$\Pi' \equiv \dfrac{\begin{array}{c}\dfrac{\dfrac{\begin{array}{c}\Lambda_1\\ \Box G_1\end{array}\ \ldots\ \begin{array}{c}\Lambda_m\\ \Box G_m\end{array}\ \begin{array}{c}\Sigma_1\\ \Box B_1\end{array}\ \ldots\ \begin{array}{c}\Sigma_n\\ \Box B_n\end{array} \qquad \begin{array}{c}\dfrac{[\Box G_1]^k\ \ldots\ [\Box G_m]^k \quad \begin{array}{c}[\Box G_1]^k\ldots[\Box G_m]^k\\ \Lambda_{m+1}\\ A\end{array}}{\Box A}\,k \quad [\Box B_1]^r\ \ldots\ [\Box B_m]^r\\ \Sigma_{n+1}\\ B\end{array}}{\Box B}\,r \qquad [\neg\Box B]^i}{\bot}\\ \Sigma_0\\ \bot\end{array}}{\Box B}\,i$$

3 Problems on reduction procedure

In this section, we indicate some problems which show that the reduction procedure does not decrease the index, or the measure for induction, of the derivation.

3.1

In the case of our subsection 2.1, which corresponds to the case (2.1) of [3], we have to notice that the assumptions of the form $\neg F$ discharged at the rule i may occur in more than one place in Π. Furthermore, there may be maximal formulas with the degree equal to that of Π other than F in the upper formulas of r. That is, there may be H_l such that $g(F) = g(H_l)$. If so is the case, the index of Π' may greater than that of Π, because the maximal formula H_l of maximal degree in Π generates plural copies of itself in Π'. Therefore, the induction does not work at this case.

3.2

In the case of our subsection 2.2, which corresponds to the case (2.2.3) of [3], we also have to notice that the assumptions of the form $\neg\Box A$ discharged at the rule i may occur in more than one place in Π. According to the definition in [3], the formula $\Box A$ which is the side-set formula of the discharged $\neg\Box A$

in Π is not a maximal formula, but the corresponding $\Box A$ in Π' must be. Thus, if the transformation from Π to Π' makes plural copies of $\Box A$, the index of Π' may be greater than that of Π. Moreover, if there is $\Box B_l$ such that $g(\Box B_l) = g(\Box A)$ in Π, and if there are more than one place of discharged $\neg\Box A$ in Π, there may be plural copies of $\Box B_l$ of maximal degree in Π'. Therefore, the induction does not work also at this case.

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