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A NOTE ON THE RATE OF CONVERGENCE IN CENTRAL LIMIT THEOREM FOR STATIONARY PROCESSES

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ABSTRACT

In this note we shall estimate the rate of convergence in central limit theorem for stationary processes. Similar results have been obtained by I. A. Ibragimov (1963) and Y. Kato (1975) in the martingale case.

INTRODUCTION

Suppose that there exists a probability measure P defined on a Borel field $\mathfrak{M}$ of sets of some space X. Space L_2 corresponds to measure P; $\|f\|$ denotes norm of a function f in L_2. If a Borel field $\mathfrak{D}$ is contained in $\mathfrak{M}$, then $H(\mathfrak{D})$ denotes Hilbert space of those function in L_2, which is measurable with respect to $\mathfrak{D}$. P_G denotes orthogonal projection onto closed subspace $G \subset L_2 = H$.

Let T be a 1-1 measure-preserving point transformation on X and $\mathfrak{M}_0$ be a Borel field such that $T^{-1}(\mathfrak{M}_0) \subset \mathfrak{M}_0$. Relation $Uf(x) = f(T^{-1}x)$ defines a unitary transformation U on H.

Spaces $\mathfrak{M}_k$, H_k, C_k, $\mathfrak{R}$ and $\mathfrak{F}$ are defined by the following relations;

$$\mathfrak{M}_k = T^k(\mathfrak{M}_0),$$
$$H_k = H(\mathfrak{M}_k),$$
$$C_k = H_k \ominus H_{k-1},$$
$$\mathfrak{R} = \{f;\ f \in H_k \ominus H_j \text{ for some } -\infty < j \leq k < \infty\},$$
$$\mathfrak{F} = \{f;\ \text{measurable wish respect to } \mathfrak{M}_k \text{ for some } -\infty < k < \infty\}.$$

I. A. Ibragimov (1963) obtained the rate of convergence in central limit theorem for bounded martingale difference sequence. Y. Kato (1975) obtained a better estimate than Ibragimov's result under some restriction. In this note we shall obtain by using Ibragimov's result the similar one for some stationary sequence. In next section, we shall obtain the rate of convergence in central limit theorem for sequence $\{U^k f;\ k=1, 2, \cdots\}$ in the case of $f \in \mathfrak{R}$. In last section, we shall obtain the similar result in the case of $f \in \mathfrak{F}$, but f satisfies the Gordin's condition.

§1. CASE OF $f \in \mathfrak{R}$

For $f \in \mathfrak{R}$, there exists a constant α such that

$$\alpha = \inf\{k;\ f \in H_k \ominus H_{-k}\}.$$

Then we obtain the Gordin's representation of f such that

$$f = h + g - Ug,$$

where

$$h = \sum_{j=-\alpha+1}^{\alpha} U^{-j} P_{C_j} f,$$

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$$g=\sum_{j=-\alpha+1}^{\alpha}\sum_{m=0}^{-j+1}U^m P c_j f.$$

We define the random variables

$$s_j^2=E\{(U^j h)^2\|\mathfrak{M}_{j-1}\},$$

and we also define the random index ν_n by using the inequalities

$$s_0^2+\cdots+s_{\nu_n-1}^2<n\leq s_0^2+\cdots+s_{\nu_n}^2.$$

Finally, we define a random variable γ_n such that

$$s_0^2+\cdots+s^2_{\nu_n-1}+\gamma_n s_{\nu_n}^2=n,$$

and we write

$$S_n(f)=f+Uf+\cdots+U^{\nu_n-1}f,$$

$$\tilde{S}_n(f)=f+Uf+\cdots+U^{\nu_n-1}f+\sqrt{\gamma_n}U^{\nu_n}f.$$

We shall often write ν, γ in place of ν_n, γ_n, respectively.

THEOREM 1: If a function f satisfies the following three conditions;

(1) $f\in\mathfrak{N}$

(2) $|f|\leq M<\infty$ with probability 1,

(3) $\sum_{j=0}^{\infty}s_j^2=+\infty$ with probability 1,

then

$$\sup_x\left|P\left\{\frac{S_n(f)}{\sqrt{n}}\leq x\right\}-\Phi(x)\right|\leq\frac{2}{\pi}\left\{\frac{24\sqrt{\alpha M}}{\sqrt[4]{n}}+\frac{5\alpha M}{\sqrt{n}}+\frac{2\alpha^2M^2}{n}\right\},$$

$$\sup_x\left|P\left\{\frac{\tilde{S}_n(f)}{\sqrt{n}}\leq x\right\}-\Phi(x)\right|\leq\frac{2}{\pi}\left\{\frac{25\sqrt{\alpha M}}{\sqrt[4]{n}}+\frac{5\alpha M}{\sqrt{n}}+\frac{2\alpha^2M^2}{n}\right\},$$

where $\Phi(x)$ is a standard normal distribution function.

PROOF: From the definitions of h and g, we have

$$|h|\leq 4\alpha M,\quad |g|\leq 4\alpha M,$$

and $\{U^k h\,;\,k=0,1,2,\cdots\}$ is a sequence of martingale differences. Therefore, it follows from Theorem 2 of I. A. Ibragimov (1963) that

$$I_1=\left|E\left\{\exp\left(\frac{it}{\sqrt{n}}S_n(h)+\frac{t^2}{2n}\sum_{j=0}^{\nu-1}s_j^2\right)\right\}-1\right|\leq e^{\frac{t^2}{2}}\left(\frac{4t\alpha M}{\sqrt{n}}+\frac{2t^2\alpha^2M^2}{n}+\frac{8t^4\alpha^4M^4}{n^2}\right) \tag{2.1}$$

Next, we shall estimate the quantity I_2;

$$I_2=\left|E\left\{\exp\left(\frac{it}{\sqrt{n}}S_n(h)+\frac{t^2}{2n}\sum_{j=0}^{\nu-1}s_j^2\right)\right\}-e^{\frac{t^2}{2}}E\left\{\exp\left(\frac{it}{\sqrt{n}}S_n(f)\right)\right\}\right|.$$

If follows from the definition of γ that

$$\begin{aligned}I_2&=\left|E\left[\exp\left(\frac{it}{\sqrt{n}}S_n(f)+\frac{t^2}{2}\right)\left\{\left(1-\exp\left(-\frac{it}{\sqrt{n}}(g-U^\nu g)-\frac{t^2}{2n}\gamma s_\nu^2\right)\right\}\right]\right|\\&=\left|E\left\{\exp\left(\frac{it}{\sqrt{n}}S_n(f)+\frac{t^2}{2}\right)\cdot\theta\left(\frac{it}{\sqrt{n}}(g-U^\nu g)+\frac{t^2}{2n}\gamma s_\nu^2\right)\right\}\right|\\&\leq e^{\frac{t^2}{2}}E\left\{\frac{t}{\sqrt{n}}|g-U^\nu g|+\frac{t^2}{2n}s^2_\nu\right\}\leq e^{\frac{t^2}{2}}\left(\frac{8\alpha Mt}{\sqrt{n}}+\frac{8\alpha^2M^2t^2}{n}\right),\end{aligned} \tag{2.2}$$

where θ is used to denote random variable whose modulus dose not exceed 1. Moreover, we have

$$I_3=|Ee^{\frac{it}{\sqrt{n}}S_n(f)}-Ee^{\frac{it}{\sqrt{n}}\tilde{S}_n(f)}|=|E\{e^{\frac{it}{\sqrt{n}}S_n(f)}(1-e^{\frac{it}{\sqrt{n}}\sqrt{\gamma}U^{\nu}f})\}|\leq\frac{Mt}{\sqrt{n}}. \qquad (2.3)$$

Then the inequalities given in the theorem are an easy consequence of the inequalities (2.1), (2.2), (2.3) and Essen's theorem (see e.g., Loeve (1955), p 285). Thus, for example, setting

$$z_n=\frac{\sqrt[4]{n}}{\sqrt{\alpha M}},$$

it follows from the inequalities (2.1), (2.2) and Essen's theorem that

$$\sup_x\left|P\left\{\frac{S_n(f)}{\sqrt{n}}\leq x\right\}-\Phi(x)\right|$$
$$\leq\frac{2}{\pi}\int_0^{z_n}\left(\frac{12\alpha M}{\sqrt{n}}+\frac{10t\alpha^2M^2}{n}+\frac{8t^3\alpha^4M^4}{n^2}\right)dt+\frac{24}{\pi}\frac{\sqrt{\alpha M}}{\sqrt[4]{n}}$$
$$\leq\frac{2}{\pi}\left\{\frac{24\sqrt{\alpha M}}{\sqrt[4]{n}}+\frac{5\alpha M}{\sqrt{n}}+\frac{2\alpha^2M^2}{n}\right\}.$$

Similarly we have

$$\sup_x\left|P\left\{\frac{\tilde{S}_n(f)}{\sqrt{n}}\leq x\right\}-\Phi(x)\right|\leq\frac{2}{\pi}\left\{\frac{25\sqrt{\alpha M}}{\sqrt[4]{n}}+\frac{5\alpha M}{\sqrt{n}}+\frac{2\alpha^2M^2}{n}\right\}.$$

The proof of Theorem 1 is now complete.

§2. CASE OF $f\in\mathfrak{F}$

We now assume that function f satisfies

$$f\in\mathfrak{F},$$
$$\sum_{k\geq n}E|P_{H_{-k}}f|=0(n^{-\alpha}),\ \alpha>0.$$

We shall define the functions

$$f_n=P_{H_{[n^{\frac{1}{2+\alpha}}]}}f-P_{H_{-[n^{\frac{1}{2+\alpha}}]}}f,$$

then we obtain the Gordin's representation of f such that

$$f=h_n+g_n-Ug_n+f-f_n,$$

where

$$h_n=\sum_{j=-[n^{\frac{1}{2+\alpha}}]+1}^{[n^{\frac{1}{2+\alpha}}]}U^{-j}P_{C_j}f_n,\quad g_n=\sum_{j=-[n^{\frac{1}{2+\alpha}}]+1}^{[n^{\frac{1}{2+\alpha}}]}\sum_{m=0}^{-j+1}U^mP_{C_j}f_n.$$

We define the random variables

$$s_{n,j}^2=E\{(U^jh_n)^2|\mathfrak{M}_{j-1}\},$$

and we also define the random index ν_n by using the inegualities

$$s_{n,0}^2+\cdots+s_{n,\nu_n-1}^2<n\leq s_{n,0}^2+\cdots+s_{n,\nu_n}^2.$$

Finally, we define a random variable γ_n such that

$$s_{n,0}^2+\cdots+s_{n,\nu_n-1}^2+\gamma_n s_{n,\nu_n}^2=n,$$

and we write

$$S_n(f)=f+Uf+\cdots+U^{\nu_n-1}f,\ \tilde{S}_n(f)=f+Uf+\cdots+U^{\nu_n-1}f+\sqrt{\gamma_n}U^{\nu_n}f.$$

THEOREM 2: If a function f satisfies the following four conditions;

(1) $f\in\mathfrak{F}$,

(2) $|f|\leq M<\infty$, with probability 1,

(3) $\sum_{k\geq n}E|P_{H_{-k}}f|=0(n^{-\alpha}),\ \alpha>0,$

(4) $\liminf_{n\to\infty} \text{ess.} \inf_{x\in X} |\sum_{k=-n+1}^{n} U^{-k}P_{C_k}f(x)|>0,$

then

$$\sup_x\left|P\left\{\frac{S_n(f)}{\sqrt{n}}\leq x\right\}-\Phi(x)\right|=0(n^{-\frac{\alpha}{4(2+\alpha)}}),\ \sup_x\left|P\left\{\frac{\widetilde{S}_n(f)}{\sqrt{n}}\leq x\right\}-\Phi(x)\right|=0(n^{-\frac{\alpha}{4(2+\alpha)}}.)$$

In proving the theorem, we shall use the next lemma.

LEMMA: If the condition (4) of Theorem 2 holds, there exists a positive number n_0 such that

$$\sup_{n\geq n_0} \text{ess.} \sup_{x\in X}\left|\frac{\nu_n(x)}{n}\right|\leq\frac{8}{\beta^2},$$

where

$$\beta=\liminf_{n\to\infty} \text{ess.} \inf_{x\in X}\left|\sum_{k=-n+1}^{n} U^{-k}P_{C_k}f(x)\right|.$$

PROOF OF LEMMA: It follows from the definition of h_n that

$$s_{n,j}^2=E\left\{\left(U^j \sum_{k=-[n^{\frac{1}{2+\alpha}}]+1}^{[n^{\frac{1}{2+\alpha}}]} U^{-k}P_{C_k}f\right)^2 \| \mathfrak{M}_{j-1}\right\}.$$

Furthermore, there exists a $n_0>0$ such that for all $n\geq n_0$

$$\text{ess.} \inf_{x\in X}\left|\sum_{k=-[n^{\frac{1}{2+\alpha}}]+1}^{[n^{\frac{1}{2+\alpha}}]} U^{-k}P_{C_k}f(x)\right|>\frac{\beta}{2}.$$

Thus, it follows that for all j and all $n\geq n_0$

$$s_{n,j}^2\geq\frac{\beta^2}{4} \quad \text{a. e.},$$

so that we have

$$\nu_n(x)\leq\left[\frac{4}{\beta^2}n\right]+1\leq\frac{8}{\beta^2}n \quad \text{a. e..}$$

This completes the proof of lemma.

We can now prove the theorem. It follows from (2.1) and (2.2) that

$$J_1=|E\{e^{\frac{it}{\sqrt{n}}S_n(f_n)}\}-e^{-\frac{t^2}{2}}|\leq\frac{12tM}{n^{\alpha/2(2+\alpha)}}+\frac{10t^2M^2}{n^{\alpha/(2+\alpha)}}+\frac{8t^4M^4}{n^{2\alpha/(2+\alpha)}}. \tag{3.1}$$

We shall now estimate the quantity J_2;

$$J_2=|E\{e^{\frac{it}{\sqrt{n}}S_n(f_n)}\}-E\{e^{\frac{it}{\sqrt{n}}S_n(f)}\}|.$$

Obviously, we have

$$J_2=\left|E\left\{\theta\frac{it}{\sqrt{n}}\left(\sum_0^{\nu_n-1}U^k(f-f_n)\right)e^{\frac{it}{\sqrt{n}}S_n(f_n)}\right\}\right|\leq\frac{t}{\sqrt{n}}E\left|\sum_0^{\nu_n-1}U^k(f-f_n)\right|, \tag{3.2}$$

where θ is used to denote random variable whose modulus dose not exceed 1.

Define

$$f_1^{(n)}=f-P_{H_{[n^{\frac{1}{2+\alpha}}]}}f,\ f_2^{(n)}=P_{H_{-[n^{\frac{1}{2+\alpha}}]}}f,$$

$$b=\inf\ \{k\ ;\ f \text{ is measurable with respect to } \mathfrak{M}_k\},$$

then we have

$$r^{(n)}(k)=E|f_2^{(n)}\cdot U^kf_2^{(n)}|\leq ME|P_{H_{-[n^{\frac{1}{2+\alpha}}]-k}}f|. \tag{3.3}$$

From (3.2), we have

$$J_2 \leq \frac{t}{\sqrt{n}} E\left|\sum_0^{\nu_n-1} U^k(f_1{}^{(n)}+f_2{}^{(n)})\right| \leq \sum_{i=1}^{2} \frac{t}{\sqrt{n}} \left\|\sum_{k=0}^{\nu_n-1} U^k f_i{}^{(n)}\right\|. \tag{3.4}$$

Since $f \in \mathfrak{F}$, we have

$$\frac{t}{\sqrt{n}} \left\|\sum_{k=0}^{\nu_n-1} U^k f_1{}^{(n)}\right\| \leq \frac{t}{\sqrt{n}} \left\|\sum_{k=0}^{\nu_n-1} |U^k f_1{}^{(n)}|\right\| \leq \frac{t}{\sqrt{n}} 2bM. \tag{3.5}$$

Furthermore, it follows from Lemma and (3.3) that for all $n \geq n_0$

$$\frac{t}{\sqrt{n}} \left\|\sum_{k=0}^{\nu_n-1} U^k f_2{}^{(n)}\right\| \leq \frac{t}{\sqrt{n}} \left\|\sum_{k=0}^{\left[\frac{8}{\beta^2}n\right]} |U^k f_2{}^{(n)}|\right\| \leq \frac{t}{\sqrt{n}} \left(\sum_{k=0}^{\left[\frac{8}{\beta^2}n\right]} \left(\left[\frac{8}{\beta^2}n\right]+1-k\right) r^{(n)}(k)\right)^{\frac{1}{2}}$$

$$\leq \frac{2\sqrt{2}\sqrt{Mt}}{\beta} \left(\sum_{k=0}^{\infty} E\left|P_{H_{-[n^{\frac{1}{2+\alpha}}]-k}} f\right|\right)^{\frac{1}{2}}. \tag{3.6}$$

Therefore, for example, setting

$$Z_n = n^{\alpha/4(2+\alpha)},$$

it follows from (3.1), (3.4), (3.5), (3.6), the condition (3) of Theorem 2 and Essen's theorem that

$$\sup_x \left|P\left\{\frac{S_n(f)}{\sqrt{n}} \leq x\right\} - \Phi(x)\right| = 0(n^{-\frac{\alpha}{4(2+\alpha)}}).$$

Similarly, we have

$$\sup_x \left|P\left\{\frac{\tilde{S}_n(f)}{\sqrt{n}} \leq x\right\} - \Phi(x)\right| = 0(n^{-\frac{\alpha}{4(2+\alpha)}}).$$

The proof of Theorem 2 is now complete.

REMARK: If the condition (3) in Theorem 2 is replased by

$$\sum_{k \geq n} E|P_{H_{-k}} f| = 0(e^{-\alpha n}), \quad \alpha > 0,$$

then we have

$$\sup_x \left|P\left\{\frac{f + \cdots + U^{\nu_n-1} f}{\sqrt{n}} \leq x\right\} - \Phi(x)\right| = 0\left(\frac{\log n}{\sqrt[4]{n}}\right),$$

where random index ν_n is defined by the same manner as those in Theorem 2 for

$$f_n = P_{H_{[\log n]}} f - P_{H_{-[\log n]}} f.$$

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