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RATES OF CONVERGENCE IN CENTRAL LIMIT THEOREM FOR A CLASS OF DEPENDENT R^k-VALUED RANDOM VARIABLES

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In this note we shall give an estimation to the rate of convergence in central limit theorem for a class of dependent R^k-valued random variables. Our method is based on Skorohod vector embedding and is the same manner as Y. Kato [15]. Similar results for independent random vectors have been obtained by many authors, for example, Rao [1], von Bahr [2], [3], Bhattacharya [4], [5], [6], [7], Sazonov [8], [9], Rotar [10] and Paulauskas [11].

Let $\{X_n=(X_{n,1}, \cdots, X_{n,k})\ ;\ n\in z^+\}$ be a sequence of R^k-valued random variables with

(1) $P\{|X_{n,i}|\leqq C\}=1$

for each $n\in z^+$ and each $i\in z_k^+$, where $z^+=\{1, 2, \cdots\}$ and $z_k^+=\{1, 2, \cdots, k\}$. For each $n\in z^+$ and each $i\in z_k^+$, let $\mathfrak{B}_{n,i}$ be a σ-algebra generated by the random variables $X_{m,j}$, $m\in z^+$, $j\in z_{i-1}^+$ and $X_{m,i}$, $m\in z_n^+$. In particular, for each $i\in z_k^+$,

$$\mathfrak{B}_{0,i}=\bigvee_{n\in Z^+}\mathfrak{B}_{n,i-1}$$

where $\mathfrak{B}_{n,0}$ is a trivial σ-algebra for each $n\in z^+$.

We define the random variables

$$s_{n,i}^2=E\{X_{n+1,i}^2\|\mathfrak{B}_{n,i}\},\quad n\in z^+\cup\{0\},\quad i\in z_k^+$$

and we also define the random indexes $\nu_{n,i}$ by the inequalities

(2) $s_{0,i}^2+\cdots+s_{\nu_{n,i}-1,i}^2<n\leqq s_{0,i}^2+\cdots+s_{\nu_{n,i},i}^2$, $n\in z^+$, $i\in z_k^+$.

Finally, we define

$$S_n=\left(\sum_{j=1}^{\nu_{n,1}}X_{j,1},\ \cdots,\ \sum_{j=1}^{\nu_{n,k}}X_{j,k}\right).$$

Throughout this note, we shall assume the following conditions:

(A) For each $i\in z_k^+$, $\{X_{n,i}, \mathfrak{B}_{n,i}\ ;\ n\in z^+\}$ is a sequence of martingale differences.

(B) For each $i\in z_k^+$, it holds that

(3) $\sum_{h=0}^{n} s_{h,i}^2 \overset{\cup}{\cap} f_i(n)$ uniformly

for some monotone increasing function $f_i(n)$ such that

(4) $\dfrac{f_i^{-1}(n)}{n}=o(\log n)$, $\dfrac{f_i(n)}{n}=0(1)$.

REMARK: The relation (3) means that

$$0<\beta_i\leqq\gamma_i<\infty,$$

where

$$\beta_i=\liminf_{n\to\infty}\ \underset{\omega}{\text{ess. inf}}\ \frac{n}{f_i(n)}\sum_{h=0}^{n}s_{h,i}^2(\omega),$$

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$$\gamma_i = \limsup_{n\to\infty} \text{ ess. } \sup_{\omega} \frac{1}{f_i(n)} \sum_{h=0}^{n} s_{h,i}^2(\omega).$$

Let Q_n denote the distribution of $\frac{1}{\sqrt{n}} S_n$ and let Φ be the standard normal distribution on R^k.

THEOREM 1: Let $\{X_n\, ;\ n\in z^+\}$ be a sequence of R^k-valued random variables satisfying the relation (1). Under assumptions (A) and (B), there exists constant M_1 such that for sufficiently large n and any Borel sets A

$$\left|Q_n(A)-\Phi(A)\right| \leqq 2k\frac{\log n}{\sqrt{n}} + \Phi((\partial A)^{\eta}),$$

where

$$\eta = Ck^{\frac{1}{2}}\frac{(\log n)^3}{\sqrt{n}} + M_1\frac{(\log n)^{\frac{3}{2}}}{\sqrt{n}}$$

The constant M_1 depends on $\gamma_1, \cdots, \gamma_k$ and C, the set ∂A is the boundary of A and $(\partial A)^{\eta}$ is the set of all points whose distances from ∂A are less than η.

In proving the theorem, we shall use Lemma 2, Lemma 3 of Kato [15] and the following lemma. The one is a Skorohod vector embedding which is a corollary of Skorohod embedding for martingale differences, see Kiefer [13] and Strassen [14].

LEMMA 1: Let $\{X_n=(X_{n,1}, \cdots, X_{n,k})\, ;\ n\in z^+\}$ be a sequence of R^k-valued random variables satisfying assumption (A) such that for each $i\in z_k^+$ and $n\in z^+$, $E\{X_{n,i}^2 \| \mathfrak{B}_{n-1,i}\}$ is defined. Then, without loss of generality, there is a sequence of independent Brownian motions $\{w_i(t)\, ;\ i\in z_k^+\}$ together with a family of non-negative random variables $\{T_{m,i}\, ;\ m\in z_n^+,\ i\in z_k^+\}$ for each $n\in z^+$ such that

$$\left(w_1\left(\sum_{j=1}^{m} T_{j,1}\right), \cdots, w_k\left(\sum_{j=1}^{m} T_{j,k}\right)\right) = \sum_{j=1}^{m} X_j \text{ a. e.}$$

for each $m\in z_n^+$. Moreover if we define

$$\mathfrak{B}_{m,i}^{(n)} = \mathfrak{B}\left\{\begin{matrix} X_{1,j}, \cdots, X_{n,j}\ j\in z_{i-1}^+ \\ X_{1,i}, \cdots, X_{m,i} \end{matrix}\right\}$$

$$\mathfrak{F}_{m,i}^{(n)} = \mathfrak{B}_{m,i}^{(n)} \vee \mathfrak{B}\left\{\begin{matrix} w_j(t),\ 0\leq t\leq \sum_{h=1}^{n} T_{h,j},\ j\in z_{i-1}^+ \\ w_i(t),\ 0\leq t\leq \sum_{h=0}^{m} T_{h,i} \end{matrix}\right\}$$

then $T_{m,i}$ is $\mathfrak{F}_{m,i}^{(n)}$-measurable, $E\{T_{m,i}\|\mathfrak{F}_{m-1,i}^{(n)}\}$ is well defined and

$$\begin{aligned} E\{T_{m,i}\|\mathfrak{F}_{m-1,i}^{(n)}\} &= E\{X_{m,i}^2\|\mathfrak{F}_{m-1,i}^{(n)}\} \\ &= E\{X_{m,i}^2\|\mathfrak{B}_{m-1,i}^{(n)}\} \text{ a. e.,} \end{aligned}$$

for each $i\in z_k^+$, $m\in z_n^+$. If h is a real number >1 and $E\{X_{m,i}^{2h}\|\mathfrak{B}_{m-1,i}^{(n)}\}$ can be defined, then $E\{T_{m,i}^h\|\mathfrak{F}_{m-1,i}^{(n)}\}$ is also well defined and

$$\begin{aligned} E\{T_{m,i}^h\|\mathfrak{F}_{m-1,i}^{(n)}\} &\leqq L_n E\{X_{m,i}^{2h}\|\mathfrak{F}_{m-1,i}^{(n)}\} \\ &\leqq L_n E\{X_{m,i}^{2h}\|\mathfrak{B}_{m-1,i}^{(n)}\} \text{ a. e.,} \end{aligned}$$

where each L_h is constant which depends only on h.

The phrase 'without loss of generality' in the above lemma is used in a same sense as Strassen's one ([14], p.333). In this note we shall assume that the new probability space satisfies the same condition as Kato's one [15].

REMARK: In the above lemma, we set $\mathfrak{B}^{(n)}_{0,i}=\mathfrak{B}^{(n)}_{0,i-1}$ for $i=2, 3, \cdots, k$ and $\mathfrak{B}^{(n)}_{0,1}$ is a trivial σ-algebra. Furthermore, we set $\mathfrak{F}_{0,i}{}^{(n)}=\mathfrak{B}_{0,i}{}^{(n)}\vee\mathfrak{B}\left\{w_j(t),\ 0\leqq t\leqq\sum_{m=1}^{n}T_{m,j},\ j\in z_i{}^{+}{}_{-1}\right\}$ for $i=2, 3, \cdots, k$ and $\mathfrak{F}_{0,1}{}^{(n)}$ is a trivial σ-algebra.

We can now prove our theorem as follows. It follow from Lemma 1 that there is a family of non-negative random variables $\{T_{m,i};\ m\in z_N{}^{+},\ i\in z_k{}^{+}\}$ such that for each $m\in z_N{}^{+}$

$$(5)\quad \frac{1}{n^{\frac{1}{2}}}(X_1+\cdots+X_m)=\left(w_1\left(\sum_{j=1}^{m}T_{j,1}\right),\ \cdots,\ w_k\left(\sum_{j=1}^{m}T_{j,k}\right)\right)\ \text{a. e.},$$

where the number N, depending only on n, will be defined as follows. From the definitions of $\mathfrak{B}^{(m)}_{j,i}$ and $\mathfrak{B}_{j,i}$, there exists a positive number $m_0(i, j, n)$ such that $m\geqq m_0(i, j, n)$ implies

$$|E\{X^2_{j+1,i}\|\mathfrak{B}^{(m)}_{j,i}\}-E\{X^2_{j+1,i}\|\mathfrak{B}_{,i}\}|<1/n^3$$

for each $j\in z^{+}\cup\{0\}$ and each $i\in z_k{}^{+}$. We define

$$N=\max\{n^2,(\max_{\substack{i\in z_k^+\\ j\in z_{n^2}{}^{+}U\{0\}}}m_0(i,\ j,\ n))\}.$$

Since it follows from assumption (B) that for sufficiently large n, $\nu_{n,i}(\omega)\leqq n^2$ a. e., we have

$$(6)\quad Q_n(A)=P\left\{\left(w_1\left(\sum_{j=1}^{\nu_{n,1}}T_{j,1}\right),\ \cdots,\ w_k\left(\sum_{j=1}^{\nu_{n,k}}T_{j,k}\right)\right)\in A\right\}$$

and

$$(7)\quad \Phi(A)=P\{(w_1(1),\ \cdots,\ w_k(1))\in A\}.$$

From the property of the stopping times $T_{j,i}$, we have

$$P\left\{(w_1(s_1),\ \cdots,\ w_k(s_k))\in A^{-\eta_1}\ \text{for some}\ (s_1,\ \cdots,\ s_k)\in \mathop{X}_{i\in z_k^+}\bigcup_{m=\nu_{n,i}}^{\nu_{n,i}+[(\log n)^3]}I_{m,i}\right\}$$
$$\leqq Q_n(A)\leqq$$
$$P\left\{(w_1(s_1),\ \cdots,\ w_k(s_k))\in A^{\eta_1}\ \text{for any}\ (s_1,\ \cdots,\ s_k)\in \mathop{X}_{i\in z_k^+}\bigcup_{m=\nu_{n,i}}^{\nu_{n,i}+[(\log n)^3]}I_{m,i}\right\}$$

where $I_{m,i}=\left[\sum_{j=1}^{m}T_{j,i},\ \sum_{j=1}^{m+1}T_{j,i}\right]$ and $\eta_1=Ck^{\frac{1}{2}}\dfrac{(\log n)^3}{n^{\frac{1}{2}}}$.

Furthermore, it follows from the definition of N, assumption (B) and Lemmas 2,3 of Kato [15] that there exist sets B_i, $i=1, 2, \cdots, k$ such that, for sufficiently large n, $P(B_i{}^c)\leqq 3/n$ and

$$\delta_{n,i}=\inf_{\omega\in B_i}\sum_{j=\nu_{n,i}+1}^{\nu_{n,i}+[(\log n)^3]}T_{j,i}>\frac{(\log n)^2}{n}.$$

Therefore, we have in the same manner as Kato's one [15]

$$(8)\quad Q_n(A)\leqq\frac{4k}{n}+P\{(\tau_1^{\frac{1}{2}}w_1(1),\ \cdots\ \tau_k^{\frac{1}{2}}w_k(1))\in A^{\eta_1}\}$$

where τ_1, τ_2, $\cdots$, τ_k are random variables satisfying the following conditions:

(Ⅰ) $\tau_1, \tau_2, \cdots, \tau_k$ are pairwise independent and are independent of all the $w_j(t)$, $j\in z_k^+$.

(Ⅱ) For each $i\in z_k^+$, $P\left\{\tau_i\in\left[\sum_{j=1}^{\nu_{n,i}}T_{j,i},\ \sum_{j=1}^{\nu_{n,i}}T_{j,i}+\delta_{n,i}\right]\right\}>1-1/n.$

From the definitions of N, $\nu_{n,i}$ and Lemma 2, Lemma 3 of Kato [15], there exist constants M_i^*, $i\in z_k^+$ such that

$$P\left\{1-M_i^*\frac{\log n}{n^{\frac{1}{2}}}\leqq\tau_i\leqq1+M_i^*\frac{\log n}{n^{\frac{1}{2}}}\right\}$$
$$>1-\frac{4}{n}-\frac{2}{n^2}f_i^{-1}\left(\frac{2}{\beta_i}n\right)\cdot\left(\exp\left\{\frac{C_1}{n}f_i^{-1}\left(\frac{2}{\beta_i}n\right)\right\}+C_2\right),$$

where $C_1=2(1+e/2)C^4$, $C_2=C^8$ and the constant M_i^* depends only on C and γ_i. Therefore, we have from (8)

$$(9)\quad Q_n(A)\leqq\frac{9k}{n}+\sum_{i=1}^{k}\frac{2}{n^2}f_i^{-1}\left(\frac{2}{\beta_i}n\right)\cdot\left(\exp\left\{\frac{C_1}{n}f_i^{-1}\left(\frac{2}{\beta_i}n\right)\right\}+C_2\right)$$
$$+P\{(w_1(1),\ \cdots,\ w_k(1))\in A^{\eta}\},$$

where

$$\eta=Ck^{\frac{1}{2}}\frac{(\log n)^3}{n^{\frac{1}{2}}}+M_1\frac{(\log n)^{\frac{3}{2}}}{n^{\frac{1}{2}}},\quad M_1=\left(\sum_{i=1}^{k}M_i^{*2}\right)^{\frac{1}{2}}.$$

In an analogous fashion, we can show that

$$(10)\quad Q_n(A)\geqq-\frac{9k}{n}-\sum_{i=1}^{k}\frac{2}{n^2}f_i^{-1}\left(\frac{2}{\beta_i}n\right)\cdot\left(\exp\left\{\frac{C_1}{n}f_i^{-1}\left(\frac{2}{\beta_i}n\right)\right\}+C_2\right)$$
$$+P\{(w_1(1),\ \cdots,\ w_k(1))\in A^{-\eta}\}.$$

Thus, it follows from (7), (9) and (10) that

$$|Q_n(A)-\Phi(A)|\leqq\frac{18k}{n}+\sum_{i=1}^{k}\frac{4}{n^2}f_i^{-1}\left(\frac{2}{\beta_i}n\right)\cdot\left(\exp\left\{\frac{C_1}{n}f_i^{-1}\left(\frac{2}{\beta_i}n\right)\right\}+C_2\right)$$
$$+P\{(w_1(1),\ \cdots,\ w_k(1)\in)\partial A)^{\eta}\},$$

so that, it follows from (4) that for sufficiently large n

$$|Q_n(A)-\Phi(A)|\leqq2k\frac{\log n}{n^{\frac{1}{2}}}+\Phi((\partial A)^{\eta}).$$

The proof of Theorem 1 is now complete.

REMARK: We can prove in the same manner that for any $\delta>0$

$$|Q_n(A)-\Phi(A)|\leqq2k\frac{\log n}{n^{\frac{1}{2}}}+\Phi((\partial A)^{\tilde{\eta}})$$

where

$$\tilde{\eta}=Ck^{\frac{1}{2}}\frac{(\log n)^{2+\delta}}{n^{\frac{1}{2}}}+\widetilde{M}^1\frac{(\log n)^{\frac{3}{2}}}{n^{\frac{1}{2}}}.$$

Let $\mathfrak{S}$ be the class of all Borel-measurable convex subsets of R^k, then B. von Bahr [2] proved the following lemma.

LEMMA 2: For all $h>0$

$$\sup_{C\in\mathfrak{S}} \Phi((\mathfrak{S}C)^h)\leqq 2^{\frac{3}{2}}\frac{\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{k}{2}\right)}h,$$

where $\Gamma(\cdot)$ is a gamma function.

As a corollary of Theorem 1 we have the following:

COROLLARY 1: Under the same conditions as Theorem 1, there exist constants M_1 and M_2 such that for sufficiently large n

$$\sup_{C\in\mathfrak{S}} |Q_n(C)-\Phi(C)|$$

$$\leqq 2k\frac{\log n}{n^{\frac{1}{2}}}+2^{\frac{3}{2}}\frac{\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{k}{2}\right)}\left\{M_1\frac{(\log n)^{\frac{3}{2}}}{n^{\frac{1}{2}}}+Ck^{\frac{1}{2}}\frac{(\log n)^3}{n^{\frac{1}{2}}}\right\}$$

$$\leqq M_2\frac{(\log n)^3}{n^{\frac{1}{2}}}.$$

where fhe constant M_2 depends only on k and M_1.

Let $k=2$. Denote by $\mathfrak{D}(m)$ the class of all Borel subsets of R^2 each having a boundary contained in some rectifiable curve of length not exceeding m. It is obvious to show that for any $D\in\mathfrak{D}(m)$ and any $h\in(0, 1)$

$$\Phi((\partial D)^h)\leqq 2(m+1)h.$$

Then we have the following corollary.

COROLLARY 2: Under the same conditions as Theorem 1, we have

$$\sup_{D\in\mathfrak{D}(m)} |Q_n(D)-\Phi(D)|$$

$$\leqq 4\frac{\log n}{n^{\frac{1}{2}}}+2(m+1)\left\{2^{\frac{1}{2}}C\frac{(\log n)^3}{n^{\frac{1}{2}}}+M_3\frac{(\log n)^{\frac{3}{2}}}{n^{\frac{1}{2}}}\right\}$$

$$\leqq M_4\frac{(\log n)^3}{n^{\frac{1}{2}}}.$$

where the constant M_3 depends only on C, γ_1 and γ_2.

There are Borel subsets A of R^k for which

(11) $\Phi((\partial A)^\varepsilon)\leqq d\varepsilon^\alpha \quad (\varepsilon>0)$

for some positive constants d and α, $\alpha>1$. Examples of such sets are affine subspaces of dimensions $k'<k-1$ (and their subsets and complements) and many other manifolds of dimensions $k'<k-1$, for which $\alpha=k-k'$. For any set A satisfying (11), we have

$$\Phi(A)=0 \text{ or } 1,$$

in particular, if A is an affine subspace of dimension $k'<k-1$, then we have

$$\Phi(A)=0.$$

THEOREM 2: Under the same conditions as Theorem 1, there exist constants M_1 and M_5 such that for sufficiently large n and $\alpha>1$ for any set A satisfying (11),

$$|Q_n(A)-\Phi(A)|$$

$$\leqq M_5\left(\frac{\log n}{n^{\frac{1}{2}}}\right)^\alpha+d\left\{Ck^{\frac{1}{2}}\frac{(\log n)^3}{n^{\frac{1}{2}}}+M_1\frac{(\log n)^{\frac{3}{2}}}{n^{\frac{1}{2}}}\right\}^\alpha,$$

where the constant M_5 depends only on k and α.

The proof of Theorem 2 (which we omit) can be obtained in a similar way as Theorem 1.

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