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The Theory of Riemann Integral

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Aestract

If we develop the theory of Riemann integral as the same method of Lebesque integral, Riemann integral is not inferior to Lebesque integral. The theory of Riemann integral will be reported in this paper.

I The Theory of Riemann Integral

Definition 1. Logically Closed Subclass

Let X be nonempty set, and P be the power set of X, that is, P is defined by $P=\{A|A\subset X\}$.

Let O be the subclass of P, which has following properties :

(i) $X\in O$

(ii) If A, $B\in O$, then $A\cap B\in O$

(iii) If $A\in O$, then $A^c\in O$

We call O "a logically closed subclass of P on X" or "a logically closed class on X."

From the properties (i) and (ii), it follows that $\Phi\in O$, and it follows from (ii) and (iii) that "if A, $B\in O$, then $A\cup B\in O$."

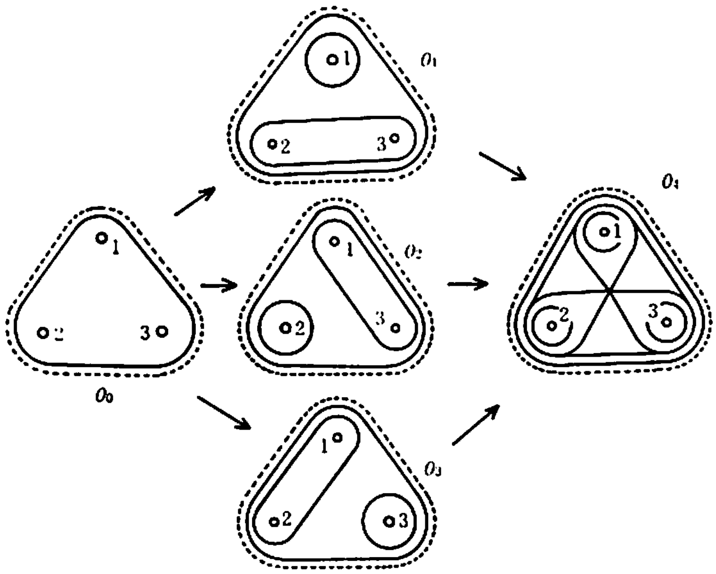

Fig. 1 $O_0\leq\begin{Bmatrix}O_1\\O_2\\O_3\end{Bmatrix}\leq O_4=P$

...indicating Φ

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Example 1

If $X=\{1, 2, 3\}$, then

$P=\{\Phi, \{1\}, \{2\}, \{3\}, \{2, 3\}, \{3, 1\}, \{1, 2\}, X\}$

P is the power set of X, but also a logically closed subclass on X, we write P as O_4.

$O_1=\{\Phi, \{1\}, \{2, 3\}, X\}$

$O_2=\{\Phi, \{2\}, \{3, 1\}, X\}$

$O_3=\{\Phi, \{3\}, \{1, 2\}, X\}$

O_1, O_2, O_3 are logically closed subclass of P on X,

$O_0=\{\Phi, X\}$

O_0 is also a logically closed subclass of P on X. These relations between O_0, O_1, O_2, O_3, O_4 are represented in Fig. 1.

Example 2

Let R be the set of all real numbers,

$X=(0, 1]=\{x|0<x\leqq 1\}\subset R$

$O_0=\{\Phi, X\}$

$O_1=\left\{\Phi, \left(0, \frac{1}{2}\right], \left(\frac{1}{2}, 1\right], X\right\}$

$O_2=\left\{\Phi ; \left(0, \frac{1}{2^2}\right], \left(\frac{1}{2^2}, \frac{2}{2^2}\right], \left(\frac{2}{2^2}, \frac{3}{2^2}\right], \left(\frac{3}{2^2}, 1\right] ; \left(0, \frac{2}{2^2}\right], \left(0, \frac{1}{2^2}\right]\cup\left(\frac{2}{2^2}, \frac{3}{2^2}\right],\right.$

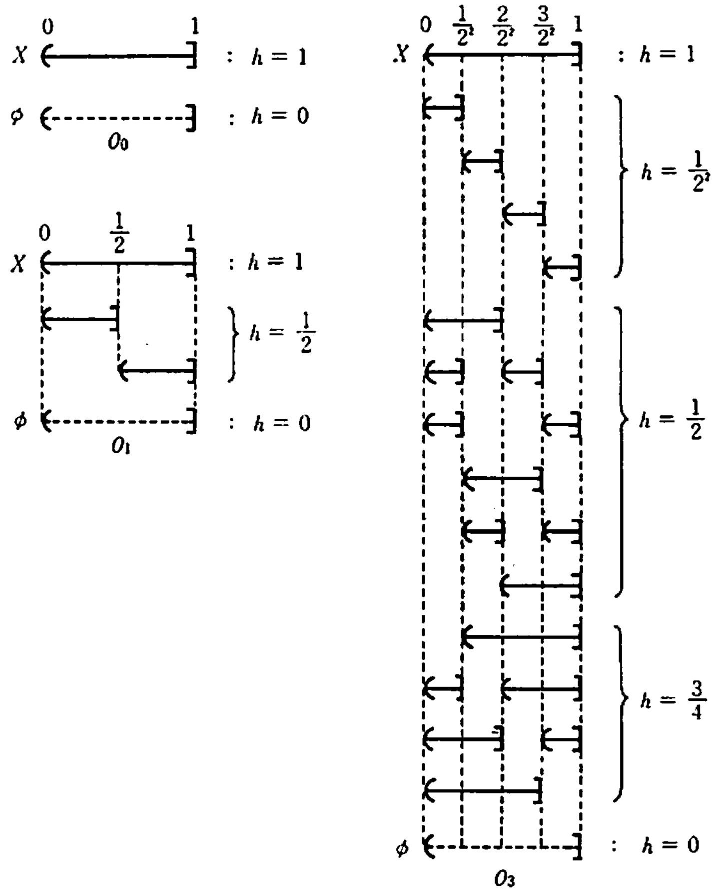

Fig. 2 O_0, O_1, O_3. Concerning h, see Example 4, which represents the length of intervals.

$\left(0, \frac{1}{2^2}\right] \cup \left(\frac{3}{2^2}, 1\right], \left(\frac{1}{2^2}, \frac{3}{2^2}\right], \left(\frac{1}{2^2}, \frac{2}{2^2}\right] \cup \left(\frac{3}{2^2}, 1\right], \left(\frac{2}{2^2}, 1\right]; \left(0, \frac{1}{2^2}\right]^c, \left(\frac{1}{2^2}, \frac{2}{2^2}\right]^c,$
$\left(\frac{2}{2^2}, \frac{3}{2^2}\right]^c, \left(\frac{3}{2^2}, 1\right]^c; X\Big\}$

....

$$O_n = \left\{\Phi, \left(0, \frac{1}{2^n}\right] \left(\frac{1}{2^n}, \frac{2}{2^n}\right] \cdots \left(\frac{2^n-1}{2^n}, 1\right], \cdots, X\right\}$$

O_1 has $2^2=4$ elements, O_2 has $2^{2^2}=16$ elements, $\cdots$, O_n has 2^{2^n} elements. O_0, O_1, O_2, $\cdots$, O_n and P are logically closed subclass of P on X,

Definition 2 Measure.

Let h be a function dlefined on a logically closed subclass O of P on X, and let $R^+ = \{x \mid x>0\} \subset R$, then h is denoted by $O \xrightarrow{h} R^+ \cup \{0\}$.

If the function h satisfies the following conditions

(i) $h(\Phi)=0$

(ii) If A, $B \in O$ and $A \cap B = \Phi$, then $h(A \cup B) = h(A) + h(B)$

then we call h "a measure on (X, O)."

Theorem 1

If h is a measure on (X, O), then the measure h has following properties:

(1) If A, $B \in O$, then $h(A)+h(B) = h(A \cup B) + h(A \cap B)$

(2) If A, $B \in O$, then $h(A \cup B) \leqq h(A) + h(B)$

(3) If C, $D \in O$, and $C \subset D$, then $h(C) \leqq h(D)$

Proof of (1)

O is closed logically and h is a measure.

$h(A \cup B) + h(A \cap B) = h(A \cup (A^c \cap B)) + h(A \cap B)$

$= h(A) + h(A^c \cap B) + h(A \cap B)$

$= h(A) + h((A^c \cap B) \cup (A \cap B))$

$= h(A) + h(B)$

(2) and (3) follow from (1). Q.E.D.

Example 3

In Example 1, let us consider (X, O_4), if we denote the number of elements in A as $|A|$, and define h as $h(A) = |A|/3$, then h is a measure on (X, O_4), with the property that $h(X)=1$.

Example 4

In Example 2, if we define h_1 as $h_1(\Phi)=0$, $h_1\left(0, \frac{1}{2}\right], = \frac{1}{2}$, $h_1\left(\frac{1}{2}, 1\right] = \frac{1}{2}$, $h(X)=1$ then h_1 is a measure on (X, O_1) representing the length of interval.

If we define h_2 as

$h_2(\Phi)=0$, $h_2\left(0, \frac{1}{2^2}\right] = \frac{1}{2^2}$, $\cdots$

$h_2\left(\left(0, \frac{1}{2^2}\right] \cup \left(\frac{3}{2^2}, 1\right]\right) = h_2\left(0, \frac{1}{2^2}\right] + h_2\left(\frac{3}{2^2}, 1\right] = \frac{1}{2}$, $\cdots$, $h_2(X)=1$.

then h_2 is a measure on (X, O_2), Which represents the legnth of interval.

As the same way, We can define h_n such that h_n is a measure on X representing the length of intervals.

Definition 3 Partition.

If I is a sulclass of O, which satisfies the following conditions

(i) $\Phi \notin I$

(ii) I is a finite cover of X : $X = \cup\{A \mid A \in I\}$ and I is finite.

(iii) I is a diojoint class : if A, $B \in O$, then $A = B$ or $A \cap B = \Phi$.

then we call I " a partition of X " or " a partion of (X, O)."

Example 5

We will consider (X, O_4) in Example 1, then we can construct five partitions of X as Fig. 3.

Example 6

In Examples 2, 4, if we define $I_0, I_1, I_2, \cdots, I_n, J$ as

$$I_0 = \{(0, 1]\}$$

$$I_1 = \left\{\left(0, \frac{1}{2}\right], \left(\frac{1}{2}, 1\right]\right\}$$

$$I_2 = \left\{\left(0, \frac{1}{2^2}\right], \left(\frac{1}{2^2}, \frac{2}{2^3}\right], \left(\frac{2}{2^2}, \frac{3}{2^2}\right], \left(\frac{3}{2^2}, 1\right]\right\}, \cdots$$

$$I_n = \left\{\left(0, \frac{1}{2^n}\right], \left(\frac{1}{2^n}, \frac{2}{2^n}\right], \cdots, \left(\frac{2^n - 1}{2^n}, 1\right]\right\}$$

$$J = \left\{\left(0, \frac{1}{2^2}\right], \left(\frac{1}{2^2}, \frac{3}{2^2}\right], \left(\frac{3}{2^2}, 1\right]\right\}$$

then $I_0, I_1, I_2, \cdots, I_n, J$ are partitions of (X, O_n).

Definition 4. Upper and Lower Integral

f is a function defined on (X, O, h), which is denoted by $X \xrightarrow{f} R$, and f is a bounded function.

This means that there exists m, $M \in R$ such that $m \leqq f(x) \leqq M$ for every $x \in X$.

For any X, O, h, I and a bounded f, we define the integral as follows

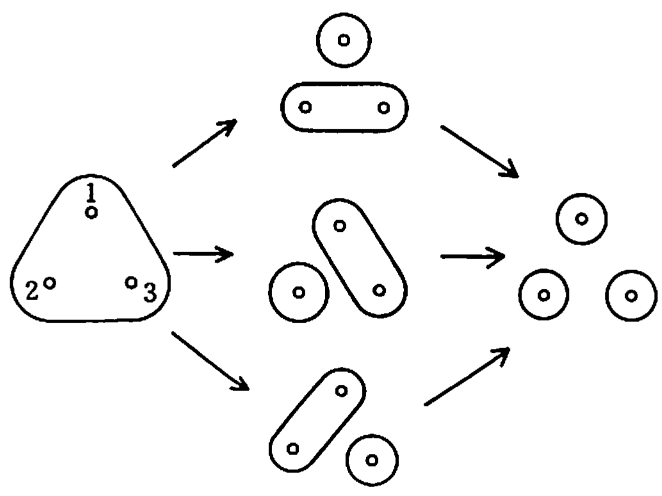

Fig. 3 Partitions of $(X. O_4)$

$I_0 = \{X\}$, $I_1 = \{\{1\}, \{2, 3\}\}$, $I_2 = \{\{2\}, \{3, 1\}\}$, $I_3 = \{\{3\}, \{1, 2\}\}$, $I_4 = \{\{1\}, \{2\}, \{3\}\}$

$\longrightarrow$ undicating refinement (See definition 5)

$M(f, A, X) = \sup f(A) = \sup \{f(x) | x \in A\}$ for each $A \in I$

$m(f, A, X) = \inf f(A) = \inf \{f(x) | x \in A\}$ for each $A \in I$

$S(f, I, X) = \Sigma\{M(f, A, X) \cdot h(A) | A \in I\}$

$s(f, I, X) = \Sigma\{m(f, A, X) \cdot h(A) | A \in I\}$

where if $h(A)$, $f(A)$ $(A \in I)$ have a unit of "cm", $S(f, I, X)$ and $s(f, I, X)$ have a unit of "$(cm)^2$," and sum Σ are taken over all $A \in I$, which is finite.

$S(f, X) = \inf \{S(f, I, X) | I$ is a partition of $X\}$

$s(f, X) = \sup \{s(f, I, X) |$I is a partion of $X\}$

where the infimum and spremum are taken over all partitions of (X, O).

We now define the upper and lower integral $\overline{\int} f$, $\underline{\int} f$ on (X, O) as

$\overline{\int} f = S(f, X)$, $\underline{\int} f = s(f, X)$

We also write the upper and lower integral

$\overline{\int}_X fdx = \overline{\int} fdx = \overline{\int} f$

$\underline{\int}_X fdx = \underline{\int} fdx = \underline{\int} f$

Example 7

In the case of (X, O_4, h), in Examples 1, 3, 5, we will define $X \xrightarrow{f} R$ as $f(1) = 1$, $f(2) = 2$, $f(3) = 3$.

For partitions I_0, I_1, I_2, I_3, I_4 in Example 5, $S(f, I, X)$ and $s(f, I, X)$ are

$S(f, I_0, X) = 3 \times 1 = 3$

$s(f, I_0, X) = 1 \times 1 = 1$

$S(f, I_1, X) = 1 \times \frac{1}{3} + 3 \times \frac{2}{3} = 2\frac{1}{3}$

$s(f, I_1, X) = 1 \times \frac{1}{3} + 2 \times \frac{2}{3} = 1\frac{2}{3}$

$S(f, I_2, X) = 2 \times \frac{1}{3} + 3 \times \frac{2}{3} = 2\frac{2}{3}$

$s(f, I_2, X) = 2 \times \frac{1}{3} + 1 \times \frac{2}{3} = 1\frac{1}{3}$

$S(f, I_3, X) = 3 \times \frac{1}{3} + 2 \times \frac{2}{3} = 2\frac{1}{3}$

$s(f, I_3, X) = 3 \times \frac{1}{3} + 1 \times \frac{2}{3} = 1\frac{2}{3}$

$S(f, I_4, X) = 1 \times \frac{1}{3} + 2 \times \frac{1}{3} + 3 \times \frac{1}{3} = 2$

$s(f, I_4, X) = 1 \times \frac{1}{3} + 2 \times \frac{1}{3} + 3 \times \frac{1}{3} = 2$

$\overline{\int} f = S(f, X) = \inf \{S(f, I_i, X) | i \in \{0, 1, 2, 3, 4\}\} = S(f, I_4 X) = 2$

$\underline{\int} f = \sup \{s(f, I_i, X) | i \in \{0, 1, 2, 3, 4\}\} = s(f, I_4, X) = 2$

In this case, the upper integral is equall to the lower integral, in such a case, f is said to be Riemann integrable.

Definition 5 Refinement

Let I and J be partitions of (X, O), "for each $B \in J$, if there exists $A \in I$ such that $B \subset A$."

then J is called a refinement of I.

In Example 5, $I \to J$ shows that J is a refinement of I.

In Eyample 6, I_1 is refinements of I_0, I_1 ; I_2 is refinements of I_0, I_1, I_2 ; $\cdots$ and so on.

Theorem 2

Let f be a bounded function defined on (X, O, h) into R. I and J are partitions of (X, O), then the following propositions hold.

(1) $s(f, I, X) \leqq S(f, I, X)$ for every I

(2) If J is a refinement of I, then

$$s(f, I, X) \leqq s(f, J, X) \leqq S(f, J, X) \leqq S(f, I, X)$$

(3) $s(f, I, X) \leqq S(f, J, X)$ for any I, J

(4) $\underline{\int} f \leqq \overline{\int} f$

Proof

(1) $m(f, A, X) = \inf f(A) = \inf \{f(x) | x \in A\}$

$\leqq \sup \{f(x) | x \in A\} = \sup f(A) = M(f, A, X)$ for every I

$s(f, I, X) = \Sigma \{m(f, A, X) \cdot h(A) | A \in I\}$

$\leqq \Sigma \{M(f, A, X) \cdot h(A) | A \in I\} = S(f, I, X)$

(2) J is a refinement of I, it follows that for each $A \in I$, there exists $M \subseteq J$ such that $A = U\{B | B \in M\}$, where M is a disjoint finite sulclass of J.

$M(f, B, X) = \sup f(B) \leqq \sup f(A) = M(f, A, X)$ for every $B \in M$

It follows from definition 2 that

$h(A) = \Sigma \{h(B) | B \in M\}$ for each $A \in I$

Hence

$\Sigma \{M(f, B, X) h(B) | B \in M\} \leqq M(f, A, X) h(A)$ for each $A \in I$

$\Sigma \{M(f, B, X) h(B) | B \in J\} \leqq \Sigma \{M(f, A, X) h(A) | A \in I\}$

$\therefore$ $S(f, J, X) \leqq S(f, I, X)$

$s(f, I, X) \leqq s(f, J, X)$ will be proved in the same way.

$s(f, J, X) \leqq S(f, J, X)$ follows from (1).

(3) If we construct $K \subseteq O$ such that

(a) If $A \in I$, $B \in J$, $A \cap B \neq \Phi$ then $A \cap B \in K$ then K becomes a partition of X and a refinement of both I and J.

It follons from (1) and (2) that

$s(f, I, X) \leqq s(f, K, X) \leqq S(f, K, X) \leqq S(f, J, X)$

(4) $s(f, I, X) \leqq S(f, J, X)$ for any I, J.

$\sup_I s(f, I, X) \leqq S(f, J, X)$

$\therefore$ $\sup_I s(f, I, X) \leqq \inf_J S(f, J, X)$

$\therefore$ $\underline{\int} f \leqq \overline{\int} f$ Q. E. D.

Definition 6 Integrable

If $\underline{\int} f = s(f, X) = S(f, X) = \overline{\int} f$, We say that f is "Riemann integrable" or "Integrable," and write $\int f$ for their common value, we also write

$$\int_X f \text{ or } \int_X f(x)dx$$

for the integral.

The following theorem concerning the existence of the integral will be obvious.

Theorem 3

A bounded function f of (X, O, h) into R is integrable if and only if, for any $n \in N_a$, there exists a partition I such that

$$S(f, I, X) < s(f, I, X) + \frac{1}{2^n}$$

where N_a indicates the set of all natural numbers.

Example 8

Let us consider Examples 2, 4, 6, where $X = (0, 1]$ and for any $n \in N_a$, logically closed subclass O_n, and the measure h_n representing the length of intervals are given.

$N = 2^n$ then the partition I_n in Example 6, is written by

$$I_n = \left\{\left(0, \frac{1}{N}\right], \left(\frac{1}{N}, \frac{2}{N}\right], \cdots, \left(\frac{N-1}{N}, 1\right]\right\}$$

For any $x \in (0, 1] = X$, we define $f(x) = x^2$. In this case, $s(f, I_n, X)$, $S(f, I_n, X)$ in definition 4, are given by

$$\begin{aligned} S(f, I_n, X) &= \frac{1}{N^3}(1^2 + 2^2 + \cdots + N^2) \\ &= \frac{1}{N^3}\frac{N(N+1)(2N+1)}{6} \\ &= \frac{1}{6}\left(2 + \frac{1}{N}\right)\left(1 + \frac{1}{N}\right) \end{aligned}$$

$$\begin{aligned} s(f, I_n, X) &= \frac{1}{N^3}(1^2 + 2^2 + \cdots + (N-1)^2) \\ &= \frac{1}{N^3}\frac{(N-1)N(2N-1)}{6} \\ &= \frac{1}{6}\left(2 - \frac{1}{N}\right)\left(1 - \frac{1}{N}\right) \end{aligned}$$

It follows that $\frac{1}{3} \leqq s(f, X) \leqq S(f, X) \leqq \frac{1}{3}$, Hence $f(x) = x^2$ is Riemann integrable on $X = (0, 1]$.

$$\int f = \int_X f = \int_X f(x)dx = \frac{1}{3}$$

The next theorem 4 will be easily proved using the proposition (a) in the proof of theorem 2.

Theorem 4

Let I, J, K are partitions of (X, O), If the notation $I \leqq J$ means that J is a refine ment of I, then the binary relation $\leqq$ directs the class of all partitions, that is

(i) For any I, $I \leqq I$ (reflexivity)

(ii) For any I, J, K, $I \leqq J$, $J \leqq K$, then $I \leqq K$ (transitivity)

(iii) For any I, J there exists a pattition K such that $I \leqq K$, $J \leqq K$ (direction)

Theorem 5

If f is a bounded function of (X, O, h) into R, then

(1) $\underline{\int} f=-\overline{\int}(-f)$ i.e. $-\underline{\int} f=\overline{\int}(-f)$ (2) $\overline{\int} f=-\underline{\int}(-f)$ i.e. $-\overline{\int} f=\underline{\int}(-f)$

Proof

(1) $m(f, A, X)=\inf f(A)=\inf \{f(x) | x \in A\}$

$=-\sup \{-f(x) | x \in A\}=-M(-f, A, X)$ for any $A \in I$,

$s(f, I, X)=\Sigma\{m(f, A, X) \cdot h(A) | A \in I\}$

$=\Sigma\{-M(-f, A, X) \cdot h(A) | A \in I\}$

$=-\Sigma\{M(-f, A, X) h(A) | A \in I\}=-S(-f, I, X)$

$\underline{\int} f=\sup_I s(f, I, X)=\sup_I (-S(-f, I, X)=-\inf_I S(-f, I, X)=-\overline{\int}(-f)$

(2) follows from (1), because

$\overline{\int} f=\overline{\int}-(-f))=-\underline{\int}(-f)$ Q.E.D.

If we use the equation (1), that f is Riemann integrable, can be expressed by

$\overline{\int} f=-\overline{\int}(-f)$

not using the definitions of m, s, $\underline{\int}$.

Theorem 6

If, f and g are bounded functions of (X, O, h) into R, then

$\underline{\int} f+\underline{\int} g \leqq \underline{\int}(f+g) \leqq \overline{\int}(f+g) \leqq \overline{\int} f+\overline{\int} g$

Proof of the right side of the expression

$M(f+g, A, X)=\sup \{f\{x)+g(x) | x \in A\}$

$\leqq \sup \{f(x) | x \in A\}+\sup \{g(x) | x \in A\}$

$=M(f, A, X)+M(g, A, X)$ for each $A \in I$

$S(f+g, I, X)=\Sigma\{M(f+g, A, X) h(A) | A \in I\}$

$\leqq \Sigma\{(M(f, A, X)+M(g, A, X)) h(A) | A \in I\}$

$=\Sigma\{M(f, A, X) h(A) | A \in I\}+\Sigma\{M(g, A, X) h(A) | A \in I\}$

$=S(f, I, X)+S(g, I, X)$

$\inf_I S(f+g, I, X) \leqq (S(f, I, X)+S(g, I, X)$

$\inf_I S(f+g, I, X) \leqq \inf_I S(f, I, X)+\inf_I S(g, I, X)$

$\therefore \overline{\int}(f+g) \leqq \overline{\int} f+\overline{\int} g$

The left side of the expression follows from the right side, because

$\overline{\int}((-f)+(-g))=\overline{\int}(-(f+g)) \leqq \overline{\int}(-f)+\overline{\int}(-g)$

$-\overline{\int}(-f)-\overline{\int}(-g) \leqq-\overline{\int}(-(f+g))$

$\underline{\int} f+\underline{\int} g \leqq \underline{\int} f+g$ by theorem 5. Q.E.D.

From theorem 7, it follows that if f and g are integrable, then $f+g$ is integrable and

$\int(f+g)=\int f+\int g$

because the left side of the expression is equall to the right side of the expression.

Theorem 7

If f is a bounded function of (X, O, h) into R, and $\alpha \in R$ then

(1) $\overline{\int} \alpha f=\alpha \overline{\int} f$ for $\alpha \geqq 0$

(2) $\underline{\int} \alpha f=\alpha \underline{\int} f$ for $\alpha \geqq 0$

(3) $\underline{\int} \alpha f=\alpha \overline{\int} f$ for $\alpha<0$

(4) $\bar{\int}\alpha f=\alpha\underline{\int}f$ for $\alpha<0$

Proof

(1) $M(\alpha f,\ A,\ X)=\sup\{\alpha f(x)\,|\,x\in A\}$

$=\alpha\sup\{f(x)\,|\,x\in A\}=\alpha\ M(f,\ A,\ X)$ for each $A\in I$

From this equation, it follows that

$S(\alpha f,\ I,\ X)=\alpha\ S(f,\ I,\ X)$

as the proof of theorem 6.

$\bar{\int}\alpha f=\sup_{I} S(\alpha f,\ I,\ X)=\alpha\ \mathrm{sip}_{I}\ S(f,\ I,\ X)=\alpha\bar{\int}f$

(2) $\bar{\int}-\alpha f=\bar{\int}\alpha(-f)=\alpha\bar{\int}(-f)$ by (1)

$\bar{\int}-\alpha f=-\underline{\int}\alpha f,\ \bar{\int}(-f)=-\underline{\int}f$ by theorem 5

$\therefore\ -\underline{\int}\alpha f=\alpha(-\underline{\int}f)=-\alpha\underline{\int}f\longrightarrow\underline{\int}\alpha f=\alpha\underline{\int}f$

(3) $If\ f\geqq 0$ then $-\bar{\int}\alpha f=-\alpha\bar{\int}f$ by (1)

$-\bar{\int}\alpha f=\underline{\int}-\alpha f$ by theorem 5

$If\ \alpha\geqq 0$ then $\underline{\int}-\alpha f=(-\alpha)\bar{\int}f$

(4) follows from (2) in the same way as (3). Q. E. D.

From this theorem, it follows that if f is integrable, then αf is integrable and $\int\alpha f=\alpha\int f$,

Theorem 8

If f and g are bounded functions of $(X,\ O,\ h)$ into R, and $f(x)\leqq g(x)$ for each $x\in X$, then

(1) $\bar{\int}f\leqq\bar{\int}g$ (2) $\underline{\int}f\leqq\underline{\int}g$

Proof of (1)

$M(f,\ A,\ X)=\sup\{f(x)\,|\,x\in A\}$

$\leqq\sup\{g(x)\,|\,x\in A\}=M(g,\ A,\ X)$ for each $A\in I$

$S(f,\ I,\ X)=\Sigma\{M(f,\ A,\ X)h(A)\,|\,A\in I\}$

$\leqq\Sigma\{M(g,\ A,\ X)h(A)\,|\,A\in I\}=S(g,\ I,\ X)$

$\inf_{I} S(f,\ I,\ X)\leqq\inf_{I} S(g,\ I,\ X)$

$\therefore\ \bar{\int}f\leqq\bar{\int}g$ Q. E. D.

Proof of (2)

$-f(x)\geqq-g(x)$ for each x, it follows from (1) that

$\bar{\int}(-g)\leqq\bar{\int}(-f),\ -\underline{\int}g\leqq-\underline{\int}f$ by theorem 5.

$\therefore\ \underline{\int}f\leqq\underline{\int}g$ Q. E. D.

If $f(x)=0$ and $0\leqq g(x)$ for every $x\in X$, then it follows that

$0\leqq\underline{\int}g\leqq\bar{\int}g$

From theorem 8, it follows that if f and g are integrable, then

$\int f\leqq\int g$.

Theorem 9

If f is a bounded function of $(X,\ O,\ h)$ into R, then

(1) $|\bar{\int}f|,\ |\underline{\int}f|\leqq\bar{\int}|f|$

(2) It is not always true to hold $|\underline{\int}f|\leqq\underline{\int}|f|$

Proof of (1)

$-|f(x)| \leqq f(x) \leqq |f(x)|$ for every $x \in X$

$-\underline{\int}|f| = \overline{\int} - |f| \leqq \overline{\int} f \leqq \overline{\int} |f|$

The first equality is followed by theorem 5, and second and third inequality are followed by theorem 8

By theorem 2, we obtain

$\underline{\int}|f| \leqq \overline{\int}|f|$ *i.e.* $-\overline{\int}|f| \leqq -\underline{\int}|f|$

Hence $-\overline{\int}|f| \leqq \overline{\int} f \leqq \overline{\int}|f| \cdots\cdots\cdots\cdots(\alpha)$

From (α), first inequality of (1) will be obtained.

Substituting -f for f in the expression (α), the second inequaly of (1) follows.

Proof of (2)

$X = (0, 1]$,

if $x \in X$ is irrational, then $f(x) = 0$,

if $x \in X$ is rational, then $f(x) = -1$.

In this case (2) does not hold, because

$\underline{\int} f = -1$, $\underline{\int}|f| = 0 < |\underline{\int} f| = 1$ Q. E. D.

The next theorem will be easily proved.

Theorem 10

If f is a bounded function of (X, O, h) into R, then there exists m, $M \in R$ such that

$m \leqq f(x) \leqq M$ for every $x \in X$ and the next inequalities hold

$mh(X) \leqq \underline{\int} f \leqq \overline{\int} f \leqq Mh(x)$

Theorem 11

Let Y be a sulset of X in (X, O, h) and $Y \in O$ then

(1) $Q_1 = \{A \cap Y | A \in O\}$, $Q_2 = \{A \cap Y^c | A \in O\}$ are logically closed subclasses on Y and on Y^c respectively.

(2) $k_1(C) = h(C)$ for $C \in Q_1$

$k_2(D) = h(D)$ for $D \in Q_2$

are measures on (Y, Q_1) and (Y, Q_2) respectively.

(3) If f is a bounded function of (X, O, h), into R, then f is a bounded function on (Y, Q_1, k_1) and also on (Y, Q_2, k_2), and f has the following properties

(α) $\overline{\int_X} f = \overline{\int_Y} f + \overline{\int_{Y^c}} f$

(β) $\underline{\int_X} f = \underline{\int_Y} f + \underline{\int_{Y^c}} f$

Proof

(1) A, $Y \in O$, $A \cap Y \in Q_1$, O hence $Q_1 \subseteq O$,

(i) $Y \in O$, $Y \cap Y = Y \in Q_1$

(ii) C, $D \in Q_1 \subseteq O \longrightarrow C = A \cap Y$, $D = B \cap Y$, A, $B \in O \longrightarrow C \cap D = (A \cap B) \cap Y \in Q_1$.

(iii) $D \in Q_1 \longrightarrow D = B \cap Y$, $B \in O \longrightarrow D^d = B^c \cap Y$ (d represents the complement of D with respect to Y), $B^c \in O \longrightarrow D^d \in Q_1$

(2) $k_1(\Phi) = 0$,

C, $D \in Q_1$, $C \cap D = \Phi \longrightarrow k_1(C \cup D) = h(C \cup D) = h(C) + h(D) = k_1(C) + k_1(D)$

(3) Proof of (α)

For a partition I of (X, O), if there exists $A \in I$ such that $A \cap Y \neq \Phi$, $A \cap Y^c \neq \Phi$, then we construct the refinement J of I such that $A \notin J$, $A \cap Y \in J$, $A \cap Y^c \in J$.

If we consider J type partitions then (α) follows.

Proofs which are not mentioned will be easy.

Theorem 12

Let Y be a subset of X in (X, O, h) and $Y \notin O$, then

(1) there exists $Z \in O$ such that $Y \subset Z$, but it is not always true that there exists $Z_0 \in O$ such that $Z_0 \subset Z$, and $Y \subset Z$.

(2) If there exists Z_0 of (1), then "if $A \in O$, $A \cap Y = \Phi$ then $A \cap Z^0 = \Phi$," that is. "if $A \in O$, $A \subset Y^c$, then $A \subset Z_0^c$"

(3) $Q = \{A \cap Y | A \in O\}$ is a logically closed subclass on Y

(4) If there exists Z_0 of (1), and if we determine k such that $k(A \cap Y) = h(A \cap Z_0)$

then k is a measure on (Y, Q)

(5) If f is a bounded function of (X, O, h) into R, then f is a bounded function on (Y, O, k) and f has following properties

(α) $\overline{\int_Y} f \leqq \overline{\int_X} f$　　(β) $\underline{\int_X} f \leqq \underline{\int_Y} f$

Proof

(1) $X \in O$ satisfies $Y \subset X$.

Let $X = (0, 1]$, $O = \{O_n | n \in N_a\}$ of Example 2, $Y = \left[\frac{1}{2}, 1\right] \notin O$, then $Z = (\alpha, 1] \in O$ for $0 \leqq \alpha < \frac{1}{2}$ and $Y = \cap\left\{(\alpha, 1] \,|\, 0 \leqq \alpha < \frac{1}{2}\right\}$, but Y does not have a minimum Z_0.

(2) If we assume that there exists $A \in O$ such that $Y \cap A = \Phi$, $Z_0 \cap A \neq \Phi$, then $A \cap Y \longrightarrow Y \subset A^c$; $Y \subset Z_0$ from (1) ;

$\therefore$ $Y \subset Z_0 \cap A^c \in O$;

$\therefore$ From (1) $Z_0 \subset Z_0 \cap A^c$ *i.e.* $Z_0 \cap (Z_0 \cap A^c)^c = Z_0 \cap (Z_0 \cup A) = Z_0 \cap A = \Phi$

which contradicts $Z_0 \cap A \neq \Phi$

(3) $A \in O \longrightarrow A \cap Y \subset Y$　　$\therefore$ $Q \subseteq$ the power set of Y

(i) $X \in O \longrightarrow X \cap Y = Y \in Q$

(ii) $C, D \in Q \longrightarrow$ there exists $A, B \in O$ such that $C = A \cap Y$, $D = B \cap Y \longrightarrow A \cap B \in O$, $C \cap D = (A \cap B) \cap Y \in Q$

(iii) $D \in Q \longrightarrow$ there exists $B \in O$ such that $D = B \cap Y \longrightarrow D^d$ indicating the complement of D with respect to Y, $D^d = B^c \cap Y \in Q$

(4) $k(A \cap Y)$ is determined uniquely by $h(A \cap Z_0)$, $A \cap Z_0 \in O$.

(i) $k(\Phi) = h(\Phi) = 0$

(ii) $C, D \in Q$, $C \cap D = \Phi \longrightarrow$ there exists $A, B \in O$ such that $C = A \cap Y$, $D = B \cap Y$, $C \cap D = (A \cap B) \cap Y = \Phi$

$\therefore$ $A \cap B \subset Y^c \longrightarrow A \cap B \subset Z_0^c$ by (2)

$k(C) = h(A \cap Z_0)$, $k(D) = h(B \cap Z_0)$, $k(C \cup D) = h((A \cap Z_0) \cup (B \cap Z_0)) = h((A \cup B) \cap Z_0)) = h((A \cap Z_0) \cup (B \cap Z_0)) = h(A \cap Z_0) + h(B \cap Z_0) = k(C) + k(D)$

(5) will be easily proved. Q.E.D.

Theorem 13

Let g be a function defined on (X, O, h) onto a set Y, *i.e.* $g(X)=Y$, and if $Q=\{D|D\subset Y,\ g^{-1}(D)\in O\}$ then

(1) Q is a logically closed sulclass on Y.

(2) if we define k by
$k(D)=h(g^{-1}(D))$ for each $D\in Q$, then k is a measure on (Y, Q).

(3) if f is a bounded function of (Y, Q, k) into R, and if $X\xrightarrow{g}Y$ and a function g is one to one, then
(α) $\overline{\int_Y} f=\overline{\int_X} f_o g$ (β) $\underline{\int_Y} f=\underline{\int_X} f_o g$

Proof

(1) $D\subset Y$ ∴ $Q\subseteq$the power set Y

(i) $Y\subset Y,\ g^{-1}(Y)=X\in O$ ∴ $Y\in Q$

(ii) $D,\ E\in Q\longrightarrow D,\ E\subset Y$ and $g^{-1}(D),\ g^{-1}(E)\in O\longrightarrow D\cap E\subset Y$ and $g^{-1}(D)\cap g^{-1}(E)=g^{-1}(D\cap E)\in O\longrightarrow D\cap E\in Q$

(iii) $E\in Q\longrightarrow D\subset Y,\ g^{-1}(E)\in O\longrightarrow E^d\subset Y$ (E^d is the complement of E with respect to Y) and $(g^{-1}(E))^c=g^{-1}(E^d)\in O\longrightarrow E^d\in Q$

(2) (i) $k(\Phi)=h(g^{-1}(\Phi))=h(\Phi)=0$

(ii) $D,\ E\in Q,\ D\cap E=\Phi\longrightarrow D,\ E\subset Y;\ g^{-1}(D),\ g^{-1}(E)\in O$, and $D\cap E=\Phi\longrightarrow$ $k(D)=h(g^{-1}(D)),\ k(E)=h(g^{-1}(E)),\ k(D\cup E)=h(g^{-1}(D\cup E))=h(g^{-1}(D)\cup g^{-1}(E)),\ g^{-1}(\Phi)=g^{-1}(D\cap E)=g^{-1}(D)\cap g^{-1}(E)=\Phi\longrightarrow k(D\cup E)=k(D)+k(E)$

(3) $g:(X,\ O,\ h)\longrightarrow(Y,\ Q,\ k)\ ;\ f:(Y,\ Q,\ k)\longrightarrow R$ and f is bounded.
$k(D)=h(g^{-1}(D))$ for each $D\in Q$.

If J is a partition of (Y, Q), then
$I=\{g^{-1}(D)\,|\,D\in J\}$ is a partition of (X, O), because $g^{-1}:\ Y\longrightarrow X$, g^{-1} is a function and one to one with the property that $g^{-1}(Y)=X$, *i.e.* g and g^{-1} are bijective functions. If $g^{-1}(D)=A$ then $g(A)=D$.

$$M(f,\ D,\ Y)=\sup\ \{f(y)\,|\,y\in D\}=\sup\ \{f(g(x))\,|\,x\in g^{-1}(D)=A\}=M(f_o g, A, X)$$

$$S(f,\ J,\ Y)=\Sigma M(f,\ D,\ Y)k(D)\,|\,D\in J\}=\Sigma\{M(f_o g,\ A,\ X)h(A)\,|\,A\in I\}$$
$$=S(f_o g,\ I,\ X)$$

$$\overline{\int_Y} f=\inf_J S(f,\ J,\ Y)=\inf_I S(f_o g,\ I,\ X)=\overline{\int_Y} f_o g$$ Q.E.D.

The next theorem will be proved as similar as theorem 13

Theorem 14

Let g be a function of X onto (Y, Q, k), where Q is a logically closed class on Y, k is a measure on (Y, Q) and $g(X)=Y$, then

(1) $O=\{g^{-1}(D)\,|\,D\in Q\}$ is a logically closed subclass on X.

(2) If we define h by
$h(g^{-1}(D))=k(D)$
then h is a measure on (X, O)

(3) if f is a bounded function of (Y, Q, k) into R and if a function g is one to

one then,

(α) $\overline{\int_Y} f = \overline{\int_X} f_o g$ (β) $\underline{\int_Y} f = \underline{\int_Y} f_o g$

II Uniform Convergence

Definition 7 Converge Uniformly

Let N_a be a set of all natural numbers, f_n, f be functions of X into R, *i.e.* $X \xrightarrow{f^n} R$, for each $n \in N_a$, $X \xrightarrow{f} R$.

A sequence $\{f_n\}$ of functions is said to "converge uniformly" to a function f if and only if

there exists a function $m: \{\varepsilon | \varepsilon > 0\} \xrightarrow{m} N_a$ Such that for any $\varepsilon > 0$, if $m(\varepsilon) < \text{n}$ then

(a) $f(x) - \varepsilon < f_n(x) < f(x) + \varepsilon$ *i.e.*

(b) $f_n(x) - \varepsilon < f(x) < f_n(x) + \varepsilon$ *i.e.*

(c) $|f_n(x) - f(x)| < \varepsilon$ for any $x \in X$.

where the value of $m(\varepsilon)$ does not depend on x.

Theorem 15

Let f_n be a bounded function of X for each $n \in N_a$, and a sequence $\{f_n\}$ converges uniformly to a function f, then

(1) f is a bounded function of X.

(2) there exists a function l

$\{\varepsilon | \varepsilon > 0\} \xrightarrow{l} N$ such that

(α) $|\overline{\int} f_n - \overline{\int} f| < \varepsilon$ for every $n > l(\varepsilon)$

(β) $|\underline{\int} f_n - \underline{\int} f| < \varepsilon$ for every $n > l(\varepsilon)$

Proof

(1) If we put $k = m(\varepsilon) + 1$, then it holds by definition 7, (*b*) that

(b′) $f_k(x) - \varepsilon < f(x) < f_k(x) + \varepsilon$, for any $x \in X$

Because f_k is a bounded function of X into R, it follows that

$\inf f_k(X) \leqq f_k(x) \leqq \sup f_k(X)$ for any $x \in X$.

Substituting this relation to (*b*′), we obtain

$\inf f_k(X) - \varepsilon < f(x) < \sup f_k(X) + \varepsilon$ for any $x \in X$.

(2) (α) It follows by theorems 6, 8 that

$\overline{\int} f_n = \overline{\int}(f_n - f + f) \leqq \overline{\int}(f_n - f) + \overline{\int} f$

$\therefore \ \overline{\int} f_n - \overline{\int} f \leqq \overline{\int}(f_n - f) \leqq \overline{\int} |f_n - f|$ (3)

If we exchang f_n for f.

$\overline{\int} f - \overline{\int} f_n \leqq \overline{\int}(f - f_n) \leqq \overline{\int} |f_n - f|$ (4)

(i) The case of $h(X) > 0$. Let l be $l(\varepsilon) = m(\varepsilon / h(X))$, then for every $n > l(\varepsilon)$,

$|f_n - f| < \varepsilon / h(X)$,

$\overline{\int} |f_n - f| < \overline{\int}(\varepsilon / h(X)) = \varepsilon$

With (3) and (4), we have

$|\overline{\int} f_n - \overline{\int} f| < \varepsilon$ for every $n > l(\varepsilon)$

(ii) The case of $h(X) = 0$.

$\bar{\int} f_n=0$ for every $x \in X$

$\bar{\int} f=0 \quad \therefore$ (α) holds.

(β) $\underline{\int} f_n=\underline{\int}(f_n-f+f) \geqq \underline{\int}(f_n-f)+\underline{\int} f$

$\underline{\int}(f_n-f)=-\bar{\int}(f-f_n)$ by theorem 5

$\therefore \underline{\int} f_n-\underline{\int} f \leqq \bar{\int}(f-f_n) \leqq \bar{\int}|f_n-f|$ ……………(3′) (3′)

If we exchange f_n for f

$\underline{\int} f-\underline{\int} f_n \leqq \bar{\int}(f_n-f) \leqq \bar{\int}|f_n-f|$ ………………(4′) (4′)

Using (3′), (4′) in places (3), (4), in proof (α) ; the proposition (β) will follow.

From this theorem, the next theorem will be easily proved.

Theorem 16

If f_n is bounded and integrable for any $n \in N_a$, and a sequence $\{f_n\}$ converges uniformly to a function f, then

(1) f is bounded and integrable

(2) there exists a function l:

$\{\varepsilon | \varepsilon>0\} \xrightarrow{l} N_a$ such that

$|\int f_n-\int f|<\varepsilon$ for every $n>l(\varepsilon)$.

III Double Integral

Theorem 17

Let O be a logically closed class on X, and h be a measure on (X, O), in the same way, Q be a logically closed class on Y, and k be a measure on (Y, Q), then

(1) if we define

$O \times Q=\{A \times B | A \in O,\ B \in Q\}$

and O and Q have more than 4 elements respectively, then $O \times Q$ is not logically closed class on $X \times Y$.

(2) let $O*Q$ be a sulclass of the power set of $X \times Y$ which has following property (α)

(α) if $O \times Q$ is defined as (1) and $K \subseteq O \times Q$ and K is a finite class, then

$\cup\{A \times B | A \times B \in K\} \in O*Q$

when this condition holds, $O*Q$ is a logically closed class on $X \times Y$.

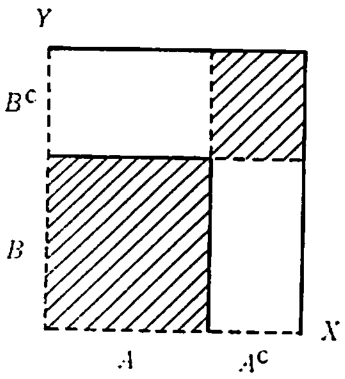

Fig. 4 Proof of (1)

Proof

(1) $O=\{\Phi,\ A,\ A^c,\ X\},\ A \neq \Phi \neq A^c$

$Q=\{\Phi,\ B,\ B^c,\ X\},\ B \neq \Phi \neq B^c$

then $A \times B,\ A^c \times B^c \in O \times Q$

but $(A \times B) \cup (A^c \times B^c) \notin O \times Q$

(2) (i) $X \in O,\ Y \in Q \longrightarrow X \times Y \in O \times Q$

$\longrightarrow X \times Y \in O*Q$

(ii) when $Z,\ W \in O*Q$ we will prove

$Z \cap W \in O*Q$

(a) $Z,\ W$ are composed by one element of $O \times Q$ respectively.

$Z=A\times B,\ W=C\times D\ ;\ A,\ C\in O\ ;\ B,\ D\in Q$

$Z\cap W=(A\times B)\cap(C\times D)=(A\cap C)\times(B\cap D)\in O*Q$

(b) Z is composed by two elements of $O\times Q$ and W is composed by one element of $O\times Q$, $A,\ M\ ,C\in O\ ;\ B,\ N,\ D\in Q$.

$Z=(A\times B)\cup(M\times N),\ \ W=C\times D$

$$\begin{aligned} Z\cap W&=((A\times B)\cup(M\times N))\cap(C\times D)\\ &=((A\times B)\cap(C\times D))\cup((M\times N)\cap(C\times D))\\ &=((A\cap C)\times(B\cap D))\cup(M\cap C)\times(N\cap D))\in O*Q \end{aligned}$$

In the same way, we can prove the case in which Z is composed by n elements of $O\times Q$ and W is composed by m elements of $O\times Q$.

(iii) $Z\in O$ we will prove $Z^c\in O$.

Z composing one element of $O\times Q$.

$Z=A\times B\longrightarrow Z^c=(A^c\times B)\cup(A\times B^c)\cup(A^c\times B^c)$.

$\therefore$ $Z^c\in O$. we can prove when Z is composed by two element of $O\times Q,\ldots,$ composed by n-elements of $O\times Q$, in the same way.

Theorem 18

Let h and k be measures on $(X,\ O)$ and $(Y,\ Q)$ respectively, and $O\times Q$, $O*Q$ be defined as theorem 17, if we define g such that

for any $A\times B\in O\times Q$, $g(A\times B)=h(A)\cdot k(B)$ then g is a function of $O\times Q$ into $R^+\cup\{0\}$, the following propositions hold

(1) for any $Z\in O*Q$, Z is expressed as

$Z=U\{A\times B\,|\,A\times B\in J\subseteq O\times Q,\ J$ is a finite disjoint class$\}$

(2) when Z is expressed as (1)

$Z=U\{A\times B\,|\,A\times B\in J\}$, we define g as $g(Z)=\Sigma\{g(A\times B)\,|\,A\times B\in J\}$

then g is a measure on $(X,\ O*Q)$.

Proof

(1) when Z consists of a element of $O\times Q$, it will be obvious, we will consider Z consists of two ellment of $O\times Q$,

$Z=(A\times B)\cup(C\times D)$ if $(A\times B)\cap(C\times D)=\Phi$, then (1) holds, if $(A\times B)\cap(C\times D)$ $\neq\Phi$ and the relation between $A\times B$ and $C\times D$ is given as Fig 5, then Z is expres-

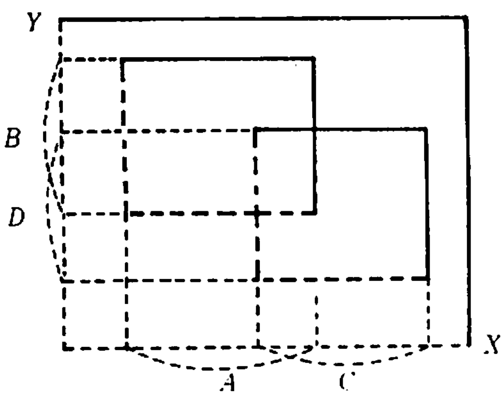

Fig. 5 Proof of (1)

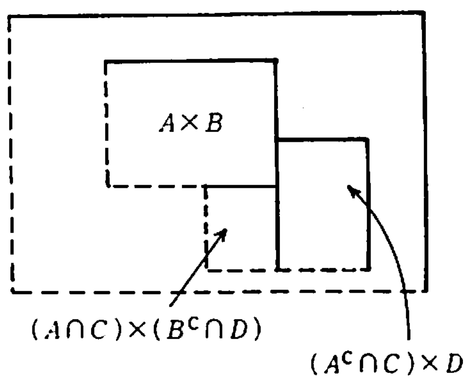

Fig. 6 Z

sed as Fig 6

$Z=(A\times B)\cup((A^c\cap C)\times D)\cup((A\cap C)\times(B^c\cap D))$

Z consists of the disjoint set.

The other case will be disposed samely.

(2) $Z\in O*Q$

when Z is disjointed in two ways such that

$Z=\cup\{A\times B|A\times B\in I\}$, $Z=U\{A\times B|A\times B\in J\}$

but it holds that

$\Sigma\{g(A\times B)|A\times B\in I\}=\Sigma\{g(A\times B)|A\times B\in J\}=g(Z)$

hence g is a function of $O*Q$ into $R^+U\{0\}$,

It will be proved for any case that

$g(\Phi)=0.$

Z, $W\in O$; $Z\cap W=\Phi$ then $g(Z\cup W)=g(Z)+g(W)$ Q. E. D.

Theorem 19

Let f be a bounded function of $X\times Y$ into R provided that (X, O, h) and (Y, Q, k) are given respectively. Let us denote $f_y(x)=f(x, y)$, then if y is fixed, f_y is a function of X into R, *i.e.* $X\xrightarrow{f_y}R$. For (X, O, h) and (Y, Q, k), we have $(X\times Y, O*Q, g)$ as theorem 18,

it follows that

(1) $\bar{\int}_Y(\bar{\int}_X f_y)\leqq\bar{\int}_{X\times Y} f$

Proof

If y is fixed, f_y is a function, $X\xrightarrow{f_y}R$.

Let I and J be partitions of (X, O), (Y, Q) respectively.

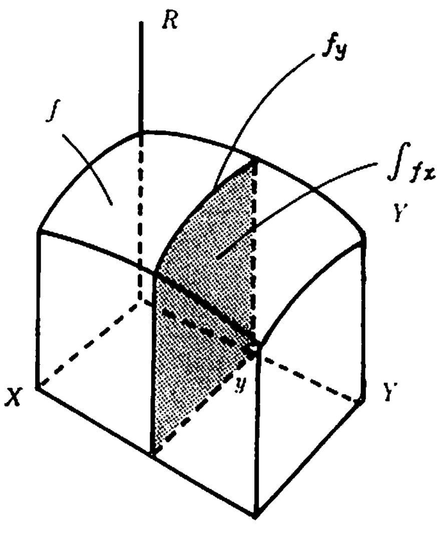

Fig. 7 Proof of (1)

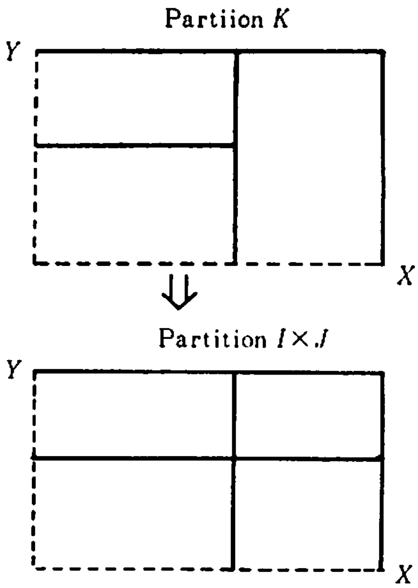

Fig. 8 Proof of (3)

$M(f_y, A, X)=\sup f_y(A) \leqq \sup f(A\times B)=M(f, A\times B, X\times Y)$ for each $A\in I$, $B\in J$.

$$\bar{\int_X} f_y \leqq S(f_y, I, X)=\sum\{M(f_y, A, X)h(A)\mid A\in I\}$$

$$\leqq \sum\{M(f, A\times B, X\times Y)h(A)\mid A\in I\} \quad \text{for each } B\in J.$$

$\bar{\int_X} f_y$ is a function such that $\bar{\int} f_y: Y \longrightarrow R$.

$$M(\bar{\int_X} f_y, B, Y)=\sup\{\bar{\int_X} f_y \mid y\in B\} \leqq \sum\{M(f, A\times B, X\times Y)h(A)\mid A\in I\}$$

for each $B\in J$

(2) $$\bar{\int_Y}(\bar{\int_X} f_y) \leqq S(\bar{\int_X} f_y, J, Y)=\sum\{M(\bar{\int_X} f_y, B, Y)k(B)\mid B\in J\}$$

$$\leqq \sum\sum\{M(f, A\times B, X\times Y)h(A)k(B)\mid A\in I, B\in J\}$$

$$=\sum\{M(f, A\times B, X\times Y)g(A\times B)\mid A\in I, B\in J\}$$

If K is a partition of $(X\times Y, O*Q)$, and the next equalities (3) hold

(3) $$\bar{\int_{X\times Y}} f=\inf_K \sum\{M(f, Z, X\times Y)g(Z)\mid Z\in K\}$$

$$=\inf_{I,J} \sum\{M(f, A\times B, X\times Y)g(A\times B)\mid A\in I, B\in J\}$$

then we have (1) from (2).

Proof of (3)

If I is a partion of (X, O), J is a partition of (Y, Q), then $I\times J=\{A\times B\mid A\in I, B\in J\}$ is a partition of $(X\times Y, O*Q)$

$$\therefore \inf_K \sum\{M(f, Z, X\times Y)g(Z)\mid Z\in K\}$$

$$\leqq \inf_{I,J} \sum\{M(f, A\times B, X\times Y)g(A\times B)\mid A\in I, B\in J\}$$

For any partition K of $(X\times Y, O*Q)$, we can construct a partition $I\times J$ of $(X\times Y, O*Q)$ which is a refininement of K, as Fig 8.

$$\therefore \inf_{I,J} \sum\{M(f, A\times B, X\times Y)g(A\times B)\mid A\in I, B\in J\}$$

$$=\inf_{I\times J} \sum\{M(f, A\times B, X\times Y)g(A\times B)\mid A\times B\in I\times J\}$$

$$\leqq \inf_K \sum\{M(f, Z, X\times Y)g(Z)\mid Z\in K\}$$

Hence, (3) follows. Q. E. D.

The next theorem will be proved as similar as theorem 19.

Theorem 20

Let f be a bounded function of $X\times Y$ into R, provided that (X, O, h) and (Y, Q, k) are given respectively. Let us denote $f_y(x)=f(x, y)$ for a fixed y, then $X \xrightarrow{f_y} R$.

For (X, O, h), (Y, Q, k), we have $(X\times Y, O*Q, g)$ as theorern 18, it follows that

$$\underline{\int_{X\times Y}} f \leqq \underline{\int_Y}(\underline{\int_X} f_y) \leqq \left\{\begin{matrix} \underline{\int_Y}(\bar{\int_X} f_y) \\ \bar{\int_Y}(\underline{\int_X} f_y) \end{matrix}\right\} \leqq \bar{\int_Y}(\bar{\int_X} f_y) \leqq \bar{\int_{X\times Y}} f$$

Example 9

Let f be a function $X\times Y$ into R, where $X=Y=(0,\ 1]$ and

$(x,\ y)\in X\times Y$, if x and y are both rational, then $f(x,\ y)=1$.

If x or y are irrational, then $f(x,\ y)=0$.

In this case

$$\overline{\int}_{X\times Y} f=1,\qquad \underline{\int}_{X\times Y} f=0$$

If y is rational

$f_y(x)=1$ for rational x.

$f_y(x)=0$ for irrational x.

$$\therefore\ \overline{\int}_X f_y=1,\quad \underline{\int}_X f_y=0$$

If y is irrational

$f_y(x)=0$ for any x.

$$\therefore\ \overline{\int}_X f_y=0,\quad \underline{\int}_X f_y=0$$

$$\overline{\int}_Y(\overline{\int}_X f=1,\quad \underline{\int}_Y(\overline{\int}_X f)=0$$

$$\underline{\int}_Y(\overline{\int}_X f_y)=0,\quad \overline{\int}_Y(\underline{\int}_X f_y)=0$$

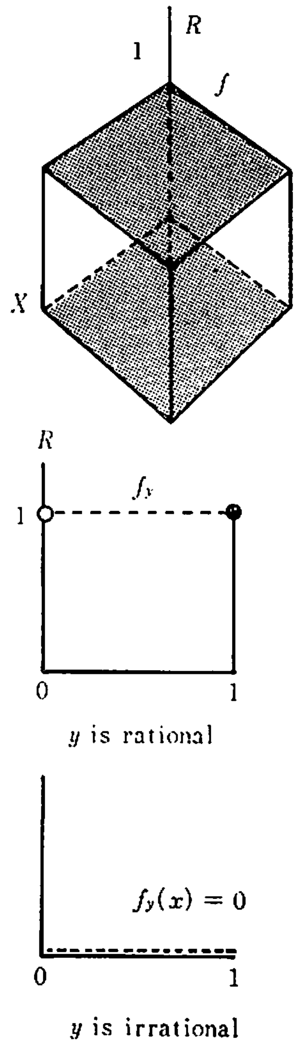

Fig. 9 f, f_y

Definition 8 Upper and Lower Integral on D.

Let D be a subset of $X\times Y$, if we define d such that

$$p=(x,\ y)\in X\times Y\longrightarrow d(p)=1 \quad \text{for } p\in D$$
$$p\in X\times Y\longrightarrow d(p)=0 \quad \text{for } p\notin D$$

then d is a function of $X\times Y$ into $\{0,\ 1\}$,

$$X\times Y\xrightarrow{d}\{0,\ 1\},$$

Let f be a bounded function of $X\times Y$ into R, then

$$p\in X\times Y\longrightarrow d\cdot f(p)=d(p)f(p)\in R$$

$d\cdot f$ is a bounded function of $X\times Y$ into R.

We define the upper and lower integral on D, respectively by

$$\overline{\int}_D f=\overline{\int}_{X\times Y} d\cdot f,\qquad \underline{\int}_D f=\underline{\int}_{X\times Y} d\cdot f$$

From the proof of theorem 20, it is sufficient that we will consider only a partition $I\times J$ instead of K which is a partition of $X\times Y$.

Theorem 21

Let D be a sulset of $(X\times Y,\ O*Q,\ g)$, if we determined a function d as same as definition 8, then the following equalities hold,

(1) $\overline{\int} d=\inf\limits_{I\times J}\ \Sigma\{g(C)\,|\,C\cap D\neq\Phi,\ C\in I\times J\}$

(2) $\underline{\int} d=\sup\limits_{I\times J}\ \Sigma\{g(C)\,|\,C\subset D,\ C\in I\times J\}$

Proof

$M(d,\ C)=1 \qquad \text{for } C\cap D\neq\Phi,\ C\in I\times J$

$M(d,\ C)=0 \qquad \text{for } C\cap D=\Phi,\ C\in I\times J$

$\therefore\ S(d,\ I\times J)=\Sigma\{M(d,\ C)g(C)\,|\,C\in I\times J\}$

$\qquad =\Sigma\{1\cdot g(C)\,|\,C\cap T\neq\Phi\}+\Sigma\{0\cdot g(C)\,|\,C\cap T=\Phi\}=\Sigma g(C)\,|\,C\cap T\neq\Phi\}$

$\therefore\ \overline{\int} d=\inf_{I\times J} S(d,\ I\times J)=\inf_{I\times J}\ \Sigma\{g(C)\,|\,C\cap T\neq\Phi,\ C\in I\times J\}$

In the same way, it follows that equality (2) holds.

$m(d,\ C)=1 \qquad \text{for } C\subset T \text{ that is } C\cap T^c=\Phi.$

$m(d,\ C)=0 \qquad \text{for } C\cap T^c\neq\Phi.$

$s(d,\ I\times J)=\Sigma\{m(d,\ C)g(C)\,|\,C\in I\times J\}$

$\qquad =\Sigma\{1\cdot g(C)\,|\,C\cap T^c=\Phi\}+\Sigma\{0\cdot g(C)\,|\,C\cap T^c\neq\Phi\}=\Sigma\{g(C)\,|\,C\cap T^c=\Phi\}$

$\therefore\ \underline{\int} d=\sup_{I\times J}\ s(d,\ I\times J)=\sup_{I\times J}\{g(C)\,|\,C\cap T^c=\Phi,\ C\in I\times J\}$ Q. E. D.

Definition 9 measurable

The set D is called a measurable set if and only if $\underline{\int} d=\overline{\int} d$

Example 10

Let $X=Y=(0,\ 1]$,

$f(p)=3$ for any $p\in X\times Y$,

$D=\{(x,\ y\,|\,\frac{1}{3}<x\leqq\frac{2}{3},\ \frac{1}{3}<y\leqq\frac{2}{3},\ x \text{ and } y \text{ are both rationals}\}$

In this case

$M(d\cdot f,\ C)=3g(C) \text{ for } C\cap D\neq\Phi$

$M(df,\ C)=0 \qquad \text{for } C\cap D=\Phi$

$m(df,\ C)=0 \qquad \text{for any } C\in I\times J$

$S(df,\ I\times J)=\Sigma\{3g(C)\,|\,C\cap D\neq\Phi,\ c\in I\times J\}=3\Sigma\{g(c)\,|\,C\cap D\neq\Phi\}$

$\overline{\int}_D f=\overline{\int}_{X\times Y} df=\inf_{I\times J}\ 3\Sigma\{g(C)\,|\,C\cap D\neq\Phi\}=3\times\frac{1}{3}\times\frac{1}{3}=\frac{1}{3}$

$\therefore\ \overline{\int}_D f=\frac{1}{3},\quad \underline{\int}_D f=\underline{\int}_{X\times Y} df=0$

Theorem 22

If f and f_1 and f_2 are bounded functions of $(X\times Y,\ O*Q,\ g)$ into R, and $D\subset X\times Y$, then

(1) (i) $\underline{\int}_D f=-\overline{\int}_D(-f)$ (ii) $\overline{\int}_D f=-\underline{\int}_D(-f)$

(2) $\underline{\int}_D f_1+\underline{\int}_D f_2\leqq\underline{\int}_D(f_1+f_2)\leqq\overline{\int}_D(f_1+f_2)\leqq\overline{\int}_D f_1+\overline{\int}_D f_2$

(3) (i) $\overline{\int}_D \lambda f=\lambda\overline{\int}_D f \qquad \text{for } \lambda\geqq 0$

(ii) $\underline{\int}_D \lambda f=\lambda\underline{\int}_D f \qquad \text{for } \lambda\geqq 0$

(iii) $\underline{\int}_D \mu f=\mu\overline{\int}_D f \qquad \text{for } \mu<0$

(iv) $\overline{\int}_D \mu f=\mu\underline{\int}_D f \qquad \text{for } \mu<0$

(4) $f_1(p)\leqq f_2(p)$ for any $p\in X\times Y$ then

(i) $\overline{\int}_D f_1\leqq\overline{\int}_D f_2$ (ii) $\underline{\int}_D f_1\leqq\underline{\int}_D f_2$

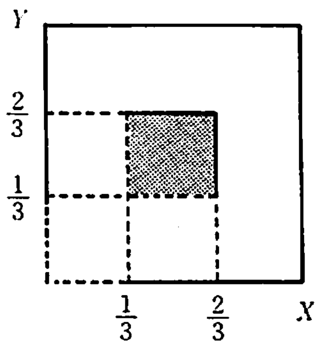

Fig. 10

Proof of (1)

$m(df, C)=\inf\{d(p)f(p)\mid p\in C\}=-\sup\{-d(p)f(p)\mid p\in C\}=-M(-df, C)$

$s(df, I\times J)=\Sigma\{m(df, C)g(C)\mid C\in I\times J\}=\Sigma\{-M(-df, C)g(C)\mid C\in I\times J\}$

$=-\Sigma\{M(-df, C)g(C)\mid C\in I\times J\}=-S(-df, I\times J)$

$\underline{\int}_D f=\underline{\int}_{X\times Y} d\cdot f=\sup_{I\times J} s(df, I\times J)=\sup_{I\times J}(-s(-df, I\times J))$

$=-\inf_{I\times J} S(-df, I\times J)=-\overline{\int}_{X\times Y}(-df)=-\overline{\int}_{X\times Y} d(-f)=-\overline{\int}_D(-f)$ (i)

(ii) is follows from (i), because

(i) $\overline{\int}_D -f_1=-\underline{\int}_D f_1,\ f=-f_1\ \ f_1=-f$

$\overline{\int}_D f=\underline{\int}_D -f$ (ii)

Proof of the right side of the expression (2)

$M(d(f_1+f_2), C)=\sup\{d(p)(f_1(p)+f_2(p))\mid p\in C\}$

$=\sup\{d(p)f_1(p)+d(p)f_2(p)\mid p\in C\}$

$\leqq\sup\{d(p)f_1(p)\mid p\in C\}+\sup\{d(p)f_2(p)\mid p\in C\}$

$=M(df_1, C)+M(df_2, C)$ for each $C\in I\times J$

$S(d(f_1+f_2), I\times J)=\Sigma\{M(d\cdot(f_1+f_2)C)g(C)\mid C\in I\times J\}$

$\leqq\Sigma\{(M(df_1, C)+M(df_2, C))g(C)\mid C\in I\times J\}$

$=\Sigma\{M(df_1, C)g(C)\mid C\in I\times J\}+\Sigma\{M(df_2C)g(C)\mid C\in I\times J\}$

$\inf_{I\times J} S(d(f_1+f_2), I\times J)\leqq S(df_1, I\times J)+S(df_2, I\times J)$

$\inf_{I\times J} S(d(f_1+f_2), I\times J)\leqq\inf_{I\times J} S(df_1, I\times J)+S(df_2, I\times J)$

$\inf_{I\times J} S(d(f_1+f_2), I\times J)\leqq\inf_{I\times J} S(df_1, I\times J)+\inf_{I\times J} S(df_2, I\times J)$

$\therefore\ \overline{\int}_{X\times Y} d(f_1+f_2)\leqq\overline{\int}_{X\times Y} df_1+\overline{\int}_{X\times Y} df_2$

$\therefore\ \overline{\int}_D(f_1+f_2)\leqq\overline{\int}_D f_1+\overline{\int}_D f_2$

The left side of the expression follows from the right side, because

$\overline{\int}_D((-f_1)+(-f_2))=\overline{\int}_D(-f_1+f_2))\leqq\overline{\int}_D(-f_1)+\overline{\int}_D(-f_2)$

$-\underline{\int}_D(f_1+f_2)\leqq-\underline{\int}_D f_1-\underline{\int}_D f_2$ using (1)

$\therefore\ \underline{\int}_D(f_1+f_2)\geqq\underline{\int}_D f_1+\underline{\int}_D f_2$

Proof of (3)

(i) $M(\lambda df, C)=\sup\{\lambda d(p)f(p)\mid p\in C\}=\lambda\sup\{d(p)f(p)\mid p\in C\}=\lambda M(df, C)$

From this equation, it follows that

$S(\lambda df, I\times J)=\lambda S(df, I\times J)$

$\overline{\int}_{X\times Y}\lambda df=\sup_{I\times J} S(\lambda df, I\times J)=\lambda\sup_{I\times J} S(df, I\times J)=\lambda\overline{\int}_{X\times Y} df$

$\therefore\ \overline{\int}_D\lambda f=\lambda\overline{\int}_D f$

(ii) $\overline{\int}_D -\lambda f=\overline{\int}_D\lambda(-f)=\lambda\overline{\int}_D -f$ by (i)

$\overline{\int}-\lambda f=-\underline{\int}\lambda f,\ \overline{\int}(-f)=-\underline{\int}f$ by (1)

$\therefore\ -\underline{\int}\lambda f=-\lambda\underline{\int}f \longrightarrow \underline{\int}\lambda f=\lambda\underline{\int}f$

(iii) $\lambda\geqq 0$ then $-\overline{\int}_D \lambda f=-\lambda\overline{\int}_D f$ by (i)

$-\overline{\int}_D \lambda f=\underline{\int}_D -\lambda f$ by (1)

$\therefore\ \underline{\int}_D -\lambda f=-\lambda\overline{\int}_D f$

(iv) If $\lambda\geqq 0$ then $-\underline{\int}_D \lambda f=-\lambda\underline{\int}_D f$ by (ii)

$-\underline{\int}_D \lambda f=\overline{\int}_D -\lambda f$ by (1)

$\therefore\ \overline{\int}_D -\lambda f=-\lambda\underline{\int}_D f$

Proof of (4), (i)

$M(df_1,\ C)=\sup\{d(p)f_1(p)\mid p\in C\}\leqq\sup\{d(p)f_2(p)\mid p\in C\}=M(df_2,\ C)$

for each $c\in I\times J$

$S(df_1,\ I\times J)=\sum\{M(df_1,\ C)g(C)\mid C\in I\times J\}\leqq\sum\{M(df_2,\ C)g(C)\mid C\in I\times J\}$

$=S(df_2,\ I\times J)$

$\therefore\ \inf_{I\times J} S(df_1,\ I\times J)\leqq\inf_{I\times J} S(df_2,\ I\times J)$

$\therefore\ \overline{\int}_D f_1\leqq\overline{\int}_D f_2$

(ii) $-f_1(p)\geqq -f_2(p)$ for each $p\in X\times Y$, it follows from (i) that

$\overline{\int}_D(-f_2)\leqq\overline{\int}_D(-f_1)$, $-\underline{\int}_D f_2\leqq -\underline{\int}_D f_1$ by (1)

$\therefore\ \underline{\int}_D f_1\leqq\underline{\int}_D f_2$ Q. E. D.

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On the lecture in the Tokyo University in 1976, the author have lectured this theory of Riemann integral as mentioned.

Mr. Shuji Saito, Department of Mathematics of the Tokyo University, who has attended the lecture, pointed ont the error which was contained in theorem 14 and in this paper the error has corrected according to his advice.

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