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Applying Non-standard Analysis to Topology

George TANAKA*

Abstruct

Abraham Robinson, an eminent English mathematician, published his treatise on "Non-sandard analysis" in 1961 and 1966. This treatise startled the world with the novel theory, and this great achievement will be said one of the greatest mathematical exploits of this century. This noted mathematician has revived the theory of monad of Leibniz as far back as three hundred years. (monad means infinitesimal) And for this reason, the theory of "Non-standard analysis" is now called together with "Infinitesimal analysis".

If we apply non-standard analysis to the usual topology, the general topology and Japanese topology, the main theorems of topology are easily derived and proved as this paper.

1. Usual Topology

Definition 1

Let us consider the universe $X=\{a, b, c, d\}$ with the order: $a<b<c<d$ in Fig. 1.

A subset A of X is not connected iff there are $x, y \in A$, $p \notin A$ such that $x<p<y$.

From this definition, the five sets in Fig. 1 are not connected, and the eleven sets in Fig. 1 are connected.

Definition 2

Let R be the set of all real numbers. It is considered that every point of R has infinitesimal length as Fig. 2.

Let x, y, z be points of R, and $x<y$. The closed interval $[x, y]=\{z | x \leqq z \leqq y\}$ and the open interval $(x, y)=\{z | x<z<y\}$ are represented as Fig. 2.

The subset A of R in Fig. 2 is not connected iff there are points $x, y \in A$, $p \notin A$ such that $x<p<y$. From this definition, it is obvious that the closed interval $[x, y]$ and the open interval (x, y) are connected, and the next theorem is immediate.

Theorem 1

The empty set ϕ, and every singleton $\{x\}$, and the set of all real numbers R, a closed interval, a open interval are connected. (Fig. 2 and Fig. 3)

Theorem 2

$\phi \neq A \subseteqq R$. If A is connected and $p \notin A$, then one of the following statements holds.

(a) The set A lies on the right side of p: $p<x$ for any $x \in A$.

(b) The set A lies on the left side of p: $x<p$ for any $x \in A$. (see Fig. 4)

Proof

If there are two points $x, y \in A$ as Fig. 2, (4), then the set A is not connected, which leads to contradiction.

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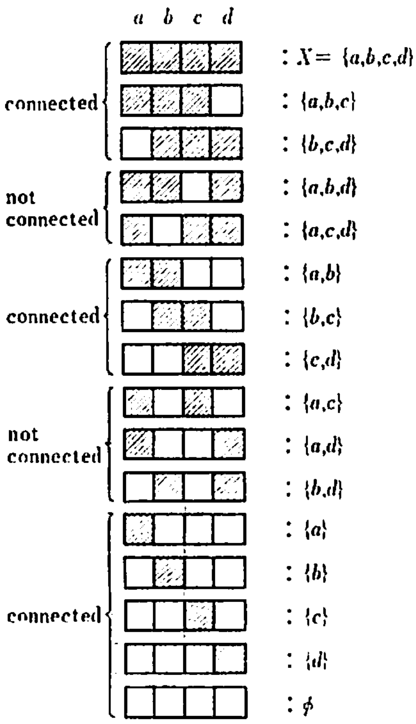


Fig. 1 Connection

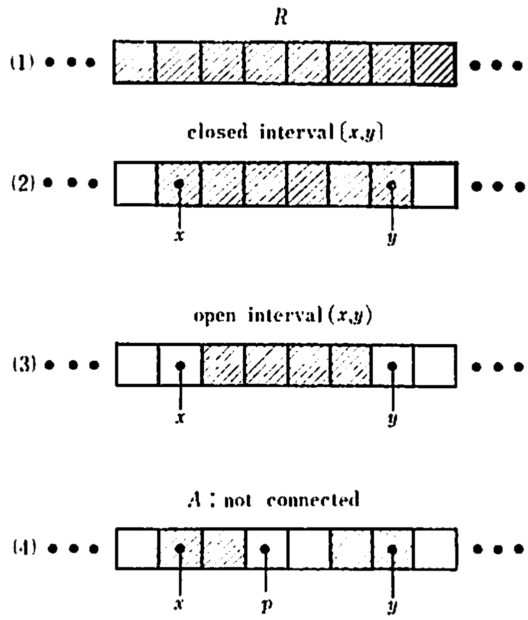


Fig. 2 Subsets of R

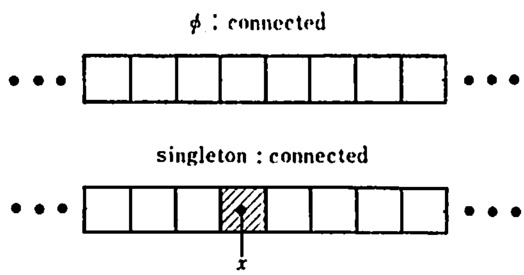


Fig. 3 ϕ and singleton

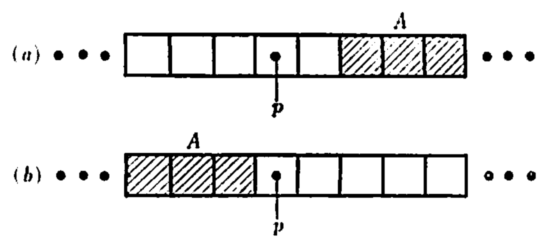


Fig. 4 The statements of theorem 2

Theorem 3

If the set A is connected in R, which contains at least two points, then A is one of the following set.

(1) The set of all real numbers: R.

(2) Closed rays: $[a, \infty)=\{x \mid a \leqq x\}$, $(-\infty, a]=\{x \mid x \leqq a\}$, $a \in R$.

(3) Open rayes: $(a, \infty)=\{x \mid a<x\}$, $(-\infty, a)=\{x \mid x<a\}$, $a \in R$.

(4) Closed intervals: $[a, b]=\{x \mid a \leqq x \leqq b\}$, $a, b \in R$ and $a<b$.

(5) Open intervals: $(a, b)=\{x \mid a<x<b\}$, $a, b \in R$ and $a<b$.

(6) Half closed (or half open) intervals: $[a, b)=\{x \mid a \leqq x<b\}$, $(a, b]=\{x \mid a<x \leqq b\}$, $a, b \in R$, $a<b$.

Proof

Question 1 Is there $p \notin A$?

Question 2 Is the set A bounded?

Question 3 Is there minimum in the set A?

Question 4 Is there maximum in the set A?

If the answer of Question 1 is "no", then $x \in A$ for any $x \in R$, it follows $A=R$.

(See Fig. 5.(1).)

The answer of Question 1 is "yes", from theorem 2, the statement (a) or (b) holds, we will assume that (a) of theorem 2 is true hereafter.

The answere of Questions.

Q1	Q2	Q3	Q4	Conclusion "A is
yes and (a)	no	yes	no	(2) in Fig. 5
〃	no	no	no	(3) 〃
〃	yes	yes	yes	(4) 〃
〃	yes	no	no	(5) 〃
〃	yes	yes	no	(6) 〃
yes and (a)	yes	no	yes	(6′) in Fig. 5"

Definition 3

We call a union of open intervals $(x, p) \cup (p, y)$ with $x<p<y$ "a open interval deleting p". This is denoted by (x, p, y), that is.

$$(x, p, y)=(x, p) \cup (p, y), \ x<p<y$$

A open interval deleting p, (x, p, y) is represented in Fig. 6.

Let A be a subset of R, a point $p \in R$ is called a cluster point of A iff every open interval deleting p has at least one point of A. (See Fig. 6)

The set of all cluster points of A is called "the derived set of A" and denoted by A'.

A point p is called "an isolated point of A" iff (i) $p \in A$ and (ii) p is not a cluster point of A, iff (i) and (ii) there is an open interval deleting p such that $(x, p, y) \cap A=\phi$. (See Fig. 7)

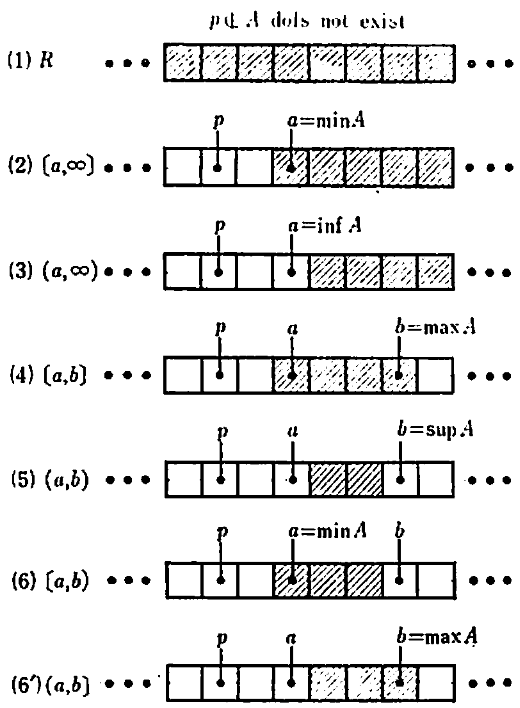


Fig. 5　Answers of Questions

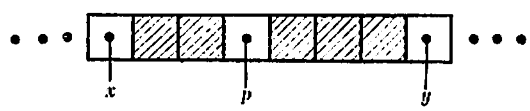


Fig. 6　(x, p, y)

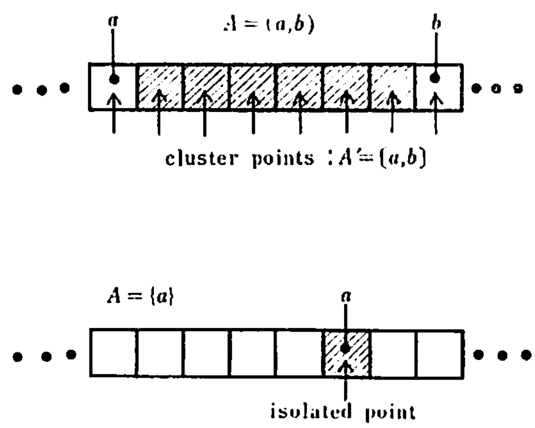


Fig. 7　Cluster point and Isolated point

If every point of A is isolated, A is called "an isolated set".

Theorem 4

If $a, b \in R$ and $a<b$; $A, B \subseteqq R$ then

(1) $(a, b)'=[a, b]$ (2) $[a, b]'=[a, b]$ (3) $\phi'=\phi$

(4) $\{a\}'=\phi$ (5) $R'=R$ (6) If $A \subseteqq B$ then $A' \subseteqq B'$

(7) $(A \cup B)'=A' \cup B'$ (8) $(A \cap B)' \subseteqq A' \cap B'$ (9) $(A')' \subseteqq A'$

Proof

(1), (4) See Fig. 7. (3), (5) will be easily proved.

(2) For any $x<a$, x is not a cluster point of $[a, b]$. For any x, there is y such that $x<y<a$ and z such that $z<x$.

$$(z, x, y) \cap [a, b]=\phi$$

Hense, x is not a cluster point of $[a, b]$. (See Fig. 8)

(6) if $(x, p, y) \cap A \neq \phi$; $A \subseteqq B$ i.e. $A=A \cap B$

$\therefore$ $(x, p, y) \cap A \cap B \neq \phi \Longrightarrow (x, p, y) \cap B \neq \phi$

that is if p is a cluster point of A, then p is a cluster point of B. (See Fig. 8)

(7) $A, B \subseteqq A \cup B$ using (6), we obtain $A', B' \subseteqq (A \cup B)'$

$\therefore$ $A' \cup B' \subseteqq (A \cup B)'$

Next we must prove that $(A \cup B)' \subseteqq A' \cup B'$.

The contrapositive is that there is p such that $p \in (A \cup B)'$, $p \notin A' \cup B'$; then $p \notin A'$,

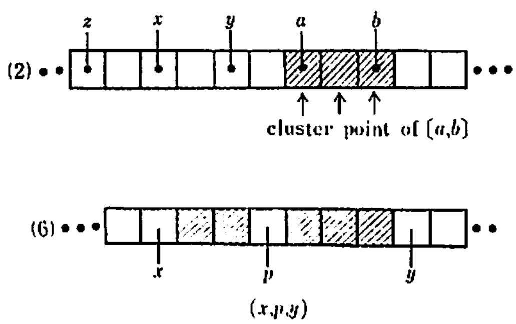


Fig. 8 (2), (6)

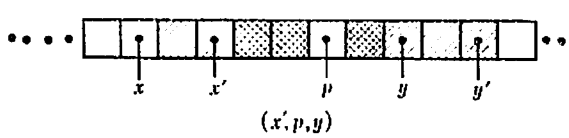
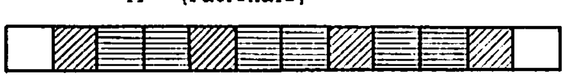


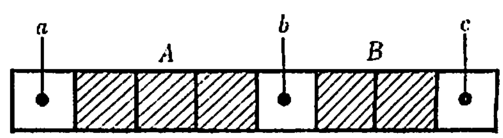
Fig. 9 (7)

$A=$ {rationals}



$B=$ {irrationals}
$(A \cap B)'=\phi'=\phi$
$A'=R$ $B'=R$
$A' \cap B'=R$

Fig. 10 (8)



$A=(a,b)$
$B=(b,c)$
$(A \cap B)'=\phi'=\phi$
$A'=[a,b]$, $B'=[b,c]$
$A' \cap B'=\{b\}$

Fig. 11 (8)

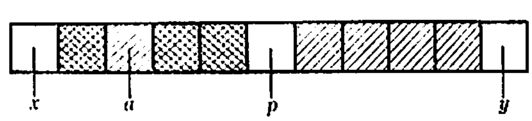


Fig. 12 (9)

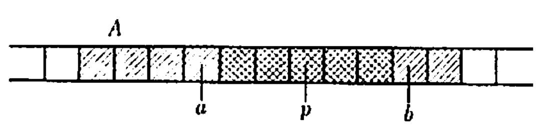


Fig. 13 $(a, b) \subseteqq A$

$p \notin B'$. There is $x<p<y$ and $x'<p<y'$ such that $x<x'$, $y<y'$,

$$(x, p, y) \cap A = \phi \quad (x', p, y') \cap B = \phi \text{ (See Fig. 9)}$$

then $(x', p, y) \cap A = \phi$ $(x', p, y) \cap B = \phi$

$(x', p, y) \cap (A \cup B) = \phi$

which contradicts that $p \in (A \cup B)'$

(8) From (6) $A \cap B \subseteqq A, B \Longrightarrow (A \cap B)' \subseteqq A', B' \Longrightarrow (A \cap B)' \subseteqq A' \cap B'$ (See Fig. 10, 11)

(9) $(A')' \subseteqq A'$

$p \in (A')'$

$(x, p, y) \cap A' \neq \phi$ for any $x<p<y$.

There is a such that

$$a \in (x, p, , y),\ a \in A' = \{z \mid (x', z, y') \cap A \neq \phi \text{ for any } x'<z<y'\}$$

if a lies on the left side of p. (See Fig. 12)

$$(x, a, p) \cap A \neq \phi$$

$\therefore$ $(x, p, y) \cap A \neq \phi$ $\therefore$ $p \in A'$

Example $A = \left\{ \frac{1}{n} \mid n \in \text{Natural number} \right\}$

$$A' = \{0\} \quad (A')' = \phi$$

Definition 4

A subset A of R is open iff for each point $p \in A$ there is an open interval (a, b); $a<p<b$ such that $(a, b) \subseteqq A$ i.e. $(a, b) \cap A^C = \phi$ as in Fig. 13.

The family of all open set in R is called the usual topology, and denoted by O.

The next theorem will be easily proved.

Theorem 5

Let O be the usual topology, then

(i) $R, \phi \in O$

(ii) If $A, B \in O$ then $A \cap B \in O$

(iii) For any $S \subseteqq O$, $\cup \{A \mid A \in S\} \in O$

2. General Topology

Definition 5

Let X be a universe and a family O of subsets of X is called a topology for X iff the next three conditions are satisfied (See Fig 14)

(i) $X \in O$

(ii) If $A, B \in O$, then $A \cap B \in O$

(iii) For any $S \subseteqq O$, $\cup \{A \mid A \in S\} \in O$

If S is the void family ϕ', then

$$\cup \{A \mid A \in \phi'\} = \phi \in O$$

The subsets of topology O are called open.

A subset M of X in closed iff A^C is open.

Fig. 15 represents the family of closed sets.

Theorem

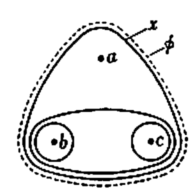


Fig. 14 $O = \{X, \{b, c\}, \{b\}, \{c\}, \phi\}$ is a topology for $X = \{a, b, c\}$

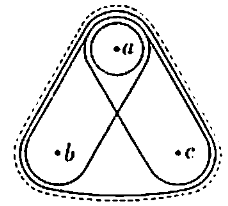


Fig. 15 $\bar{O}=\{\phi, \{a\}, \{ab\}, \{ac\} X\}$ is closed family of O in Fig 14.

If $\bar{O}$ is the closed family of *a* topology for X, then

(1) $\phi\in\bar{O}$

(2) If $A, B\in\bar{O}$, then $A\cup B\in\bar{O}$

(3) For any $T\subseteq\bar{O}$, $\cap\{A|A\in T\}\in\bar{O}$

(If T is the void family ϕ', then $\cap\{A|A\in\phi'\}=X\in\bar{O}$. Hence, X and ϕ are both open and closed)

Conversely, if *a* family $\bar{O}$ satisfies above conditions and if we define O as the complements of sets of $\bar{O}$, then O is *a* topology for X.

This theorem will easily be proved by De Morgan's rule with contrasting Fig. 14 and Fig. 15.

Definition

Let M be *a* subset of X and A be *a* open set of O, the interior of M and M^C are defined by the next equations.

Interior of $M=M^o=\cup\{A|A\subseteq M\}=\cup\{A|A\cap M^C=\phi\}$ (1)

Interior of $M^C=M^{Co}=\cup\{A|A\subseteq M^C\}=\cup\{A|A\cap M=\phi\}$ (2)

and closure of M and M^C are defined by

Closure of $M=M^-=M^{CoC}$ (3)

Closure of $M^C=M^{C-}=M^{CCoC}=M^{oC}$ (4)

Theorem 6

(1) $X^o=X$ (2) $M^o\subseteq M$ (3) $(M\cap N)^o=M^o\cap N^o$ (4) $M^{oo}=M^o$

(1′) $\phi^-=\phi$ (2′) $M\subseteq M^-$ (3′) $(M\cup N)^-=M^-\cup N^-$ (4′) $M^{--}=M^-$

Proof (1),(1′) $A\in O$, $X\in O$, $X^o=\{A|A\subseteq X\}=X$

$\phi^{CoC}=X^{oC}=X^C=\phi$

(2) $A\in O$, $M^o=\cup\{A|A\subseteq M\}\subseteq M$

(2′) $M^{Co}\subseteq M^C$ from (2), $\therefore$ $M\subseteq M^{CoC}=M^-$

(3) $(M\cap N)^o=\cup\{A|A\cap(M\cap N)^C=\phi\}=\cup\{A|A\cap(M^C\cup N^C)=\phi\}$

$=\cup\{A|(A\cap M^C)\cup(A\cap N^C)=\phi\}$

$=\cup\{A|A\cap M^C=\phi$ and $A\cap N^C=\phi\}=M^o\cap N^o$ for $A\in O$

(3′) $(M\cup N)^-=(M\cup N)^{CoC}=(M^C\cap N^C)^{oC}=(M^{Co}\cap N^{Co})^C$

$=M^{CoC}\cup N^{CoC}=M^-\cup N^-$

(4) $M^{oo}\subseteq M^o$ from (2). $M^o=\cup\{A|A\subseteq M\}\in O$

$\therefore$ $M^{oo}=\cup\{A|A\subseteq M^o\}=M^o$

(4′) $M^{--}=M^{CoCCoC}=M^{CooC}=M^{CoC}=M^-$

Theorem 7

If O and O' are topologies for X then $O\cap O'$ is *a* topology for X, but $O\cup O'$ may not be *a* topology for X.

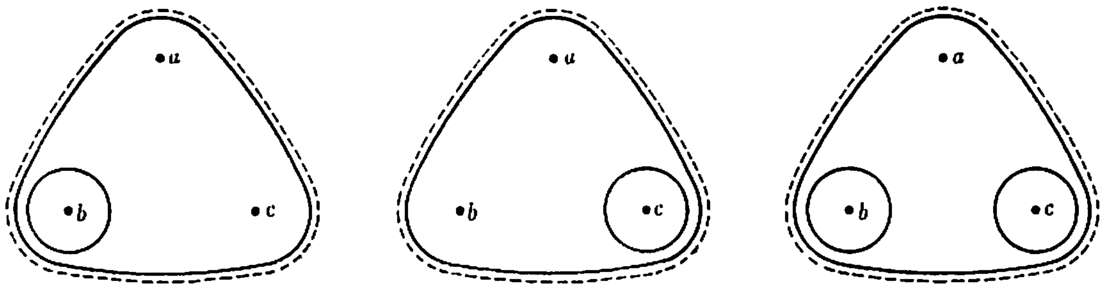
Proof

$X\in O, O'\Longrightarrow X\in O\cap O'$

$A, B\in O\cap O'\Longrightarrow A, B\in O, O'\Longrightarrow A\cap B\in O, O'\Longrightarrow A\cap B\in O\cap O'$

$S\subseteq O\cap O'\Longrightarrow S\subseteq O, O'\Longrightarrow\cup\{A|A\in S\}\subseteq O, O'\Longrightarrow\cup\{A|A\in S\}\in O\cap O'$.

The later part of the theorem follows from Fig. 16.



O is a topology for X O' is a topology for X $O \cup O'$ is not a topology for X

Fig. 16 The later part of theorem 7

3. Japanese family in Edo era

Let X be a universe and O be a subfamily of $P=\{A \mid A \subseteq X\}$. We call O the Japanese family in Edo era, iff the following three or four conditions are satisfied,

(i) $X \in O$, $\phi \notin O$

(ii) If $A, B \in O$ and $A \cap B \neq \phi$, then $A \subseteq B$ or $B \subseteq A$.

(iii) There is a unique element $a \in X$ such that if $A \in O$ and $A \neq X$ then $a \notin A$.

The condition (ii) means prohibition of "Nisoku-no-waraji". This is Japanese peculiar condition.

If $x \in X$, $x \neq a$, then "a" of the condition (iii) is called "x's Tono" and X is called "a's Nawabari" the sometimes "x's Amae to Tono a".

Let Y be a non-empty subset of X and $Y \in O$, and $O'=\{A \cap Y \mid A \in O\} \subseteq O$ then next conditions hold.

(1) $Y \in O'$, $\phi \notin O'$

(2) If M, $N \in O'$ and $M \cap N \neq \phi$, then $M \subseteq N$ or $N \subseteq M$

(iv) There is a unique element $\alpha \in Y$ such that if $M \in O'$ and $M \neq Y$, then $\alpha \notin M$

The conditions (1) and (2) will be easily proved from the definition of O' and the conditions (i) and (ii).

The condition (iv) is an additional conditions of (i), (ii), (iii), and we will assume (iv) holds for any $Y \in O$.

If $y \in Y$, $y \neq \alpha$, then "α" of the condition (iv) is called "y's Shujin" and Y is called "α's Miuchi" and also "y's Amae to Shujin α".

Theorem 8

Let O be a family of subsets of X which satisfies the conditions (i), (ii), (iii) and (iv) for each $Y \in O$, and let $\bar{O}$ be the complements of subsets of O, and let $\bar{O}'$ be the complements of subsets of O',

$$\bar{O}=\{A^c \mid A \in O\} \qquad \bar{O}'=\{M^c \mid M \in O'\}$$

then $\bar{O}$ satisfies next four conditions.

(1) $\phi \in \bar{O}$, $X \notin \bar{O}$

(2) If A, $B \in \bar{O}$ and $A \cup B \neq X$, then $A \subseteq B$ or $B \subseteq A$.

(3) There is a unique element $a \in X$ such that if $A \in \bar{O}$ and $A \neq \phi$, then $a \in A$.

(4) For each $Y \in O$, there is a unique element $\alpha \in Y$ such that if $M \in \bar{O}'$ and $M \neq \phi$ then $\alpha \in M$.

$\bar{O}$ is called "Sotomono" of O, and $\bar{O}'$ is called "Sotomono" of O',

Proof of (2) If A, $B \in \bar{O}$ and $A \cup B \neq X$ iff A^c, $B^c \in O$ and $A^c \cap B^c \neq \phi$ then $A^c \subseteq B^c$ or $B^c \subseteq A^c$ by (ii), iff $A \subseteq B$ or $B \subseteq A$.

(3) There is Tono $a \in X$ such that $A \in O$ and $A \neq X$ then $a \notin A$ iff $A^c \in \bar{O}$ and $A^c \neq \phi$ then $a \in A^c$.

(4) is the same as (3).

Examples.

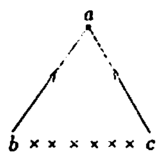
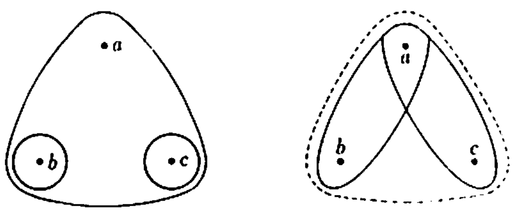
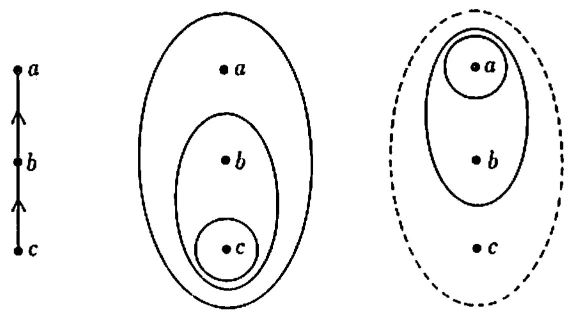


Fig. 17 × × × mean the competitive spirits between b and c.
→— means the direction of Amae of b and c to Tono a.



O : Nawabari　　$\bar{O}$: Sotomono

Fig. 18 Tono is safety and O is dangerous



O : Nawabari　　$\bar{O}$: Sotomono

Fig. 19 Tono is dangerous and O is safety

Acknowledgement

On the lecture in the Tokyo University in 1979, the auther have lectured the contents of this paper. Mr. Akio Tsumura, Department of Science of the Tokyo University, who has attended the lecture, pointed out the error which was contained in theorem 8 and in this paper the error has corrected according his advice.

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