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Integration on H-Field

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Abstract

In this paper, we use Prof. Yasusi Hattori's Field or H-Field as the base of Integration and construct the most general integral theory.

1. H-Field

Let A be a subset of X, and H be a subclass of $P=\{B;\ B\subseteqq X\}$.

$\Delta=\{A_1, A_2,\cdots, A_n\}$ is said to be a partition of A belonging to H iff

(i) For each $i\in\{1, 2,\cdots, n\}$, $A_i\in H$

(ii) $A=\bigcup\limits_{i=1}^{n} A_i$

(iii) For each $i, j\in\{1, 2,\cdots, n\}$, $i\neq j$; $A_i\cap A_j=\phi$

Definition 1

H is called a H-Field iff

(1) $\phi\in H$

(2) For each $A, B\in H$, $A\cap B\in H$

(3) For each $A, B\in H$, there is a partition of $A\sim B=A\cap B^c$ belonging to H.

Theorem 1

Let H be a H-Field and $\Delta=\{A_1, A_2,\cdots, A_m\}$ be a partition of A belonging to H, $\Delta'=\{B_1, B_2,\cdots, B_n\}$ be a partition of B; then there is a partition of $A\cap B$ belonging to H,

Proof

We *will* prove this theorem in the case of $m=2$, $n=1$.

From the definition of a partition,

$$\Delta=\{A_1, A_2\},\ \Delta'=\{B\},\ A=A_1\cup A_2$$
$$A_1\cap A_2=\phi,\quad A_1, A_2, B\in H$$
$$\therefore\ A\cap B=(A_1\cap B)\cup(A_2\cap B)$$
$$(A_1\cap B)\cap(A_2\cap B)=\phi,\quad A_1\cap B,\ A_2\cap B\in H$$

Hence $\Delta''=\{A_1\cap B, A_2\cap B\}$ is a partition of $A\cap B$ belonging to H.

Example 1

Let $X=\{a, b\}$, $P=\{X, \{a\}, \{b\}, \phi\}$ and $O=\{X, \{a\}, \phi\}$ (Fig. 1), in this case, $\phi\in O$ and O is closed by the operation of "$\cap$". Hence the conditions (1) and (2) are satisfied, but the condition (3) is not satisfied. Because there is no partition of $X\sim\{a\}=\{b\}$ belonging to O, O is an

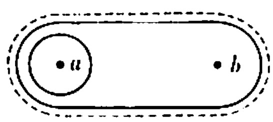

Fig. 1 The representation of $O=\{X, \{a\}, \phi\}$

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example which is not a H-Field.

Theorem 2

Let H be a H-Field, if $A_1, A_2 \in H$, then there is a partition $\Delta=\{B_1, B_2, \cdots, B_n\}$ of $A_1 \cup A_2$ belonging to H such that

$$B_i \subseteqq A_1 \text{ or } B_i \subseteqq A_2 \text{ for any } i \in \{1, 2, \cdots, n\}$$

Proof

$$A_1 \cup A_2 = A_1 \cup (A_2 \sim A_1)$$

There is a partition $\Delta'=\{C_1, C_2, \cdots, C_m\}$ of $A_2 \sim A_1$ belonging to H, then

$$\Delta=\{A_1, C_1, C_2, \cdots, C_m\}$$

is a partition of $A_1 \cup A_2$ belonging to H Such that

$$A_1 \subseteqq A_1 \; C_i \subseteqq A_2 \text{ for any } i \in \{1, 2, \cdots, m\}.$$

2. Measure μ on H

Definition 2

Let H be a H-Field and μ be a function of H into non-negative reals which is denoted by $R \sim R^-$, i.e.

$$H \xrightarrow{\mu} R \sim R^-$$

μ is called a measure on H iff for any $A \in H$ and each partition $\Delta=\{A_1, A_2, \cdots, A_n\}$ of A belonging to H,

$$\mu(A)=\mu(A_1)+\mu(A_2)+\cdots+\mu(A_n),$$

(Additivity).

In this case, $\mu(A)$ is said to be μ-measure of A and $A \in H$ is called μ-measurable set.

Theorom 3

Let H be a H-Field and μ be a measure on H, the next propositions are satisfied

(1) $\mu(\phi)=0$

(2) If $A, B \in H$, $A \subseteqq B$, then $\mu(A) \leqq \mu(B)$.

(3) If $A, B, A \cup B \in H$, then

$$\mu(A)+\mu(B)=\mu(A \cup B)+\mu(A \cap B)$$

Proof

(1) $\phi \in H$ by the definition 1(1).

$\phi \cup \phi = \phi$ is a partition of ϕ belonging to H.

$$\therefore \; \mu(\phi)+\mu(\phi)=\mu(\phi \cup \phi)=\mu(\phi)$$

$$\therefore \; \mu(\phi)=0$$

(2) $B=A \cup (B \sim A)$

There is a partition $\Delta=\{C_1, \cdots, C_n\}$ of $B \sim A$ belonging to H.

$$\Delta'=\{A, C_1, \cdots, C_n\}$$

is a partition of B belonging to H.

$$\therefore \; \mu(B)=\mu(A)+\mu(C_1)+\cdots+\mu(C_n)$$

$$\therefore \; \mu(A) \leqq \mu(B)$$

(3) There is a partition $\Delta=\{C_1,\cdots, C_n\}$ of

$$A\cup B\sim A=B\sim A=B\sim(A\cap B)$$

belonging to H.

$$\therefore\ \mu(A\cup B)=\mu(A)+\mu(C_1)+\cdots+\mu(C_n)$$

$$\therefore\ \mu(B)=\mu(A\cap B)+\mu(C_1)+\cdots+\mu(Cn)$$

From these equations, it follows that

$$\mu(A\cup B)+\mu(A\cap B)=\mu(A)+\mu(B).$$

3. Upper and Lower Integrals

Definition 3

Let H be a H-Field, μ be a measuse on H, f be a bounded function of a μ-measurable set A into R, (Reals), and $\Delta=\{A_1,\cdots, A_n\}$ be a partition of A belonging to H.

"Sum in Excess: S_Δ", "Sum in Shortage: s_Δ", "Upper Integral: S", "Lower Integral: s", are defined by the next equations

$$S_\Delta=\sum_{i=1}^{n}\sup f(Ai)\mu(A_i)\qquad \text{(Sum in Excess)}$$

$$s_\Delta=\sum_{i=1}^{n}\inf f(A_i)\mu(A_i)\qquad \text{(Sum in Shortage)}$$

$$\overline{\int_A} f\,\alpha\mu=S=\inf_\Delta S_\Delta\qquad \text{(Upper Integral)}$$

$$\underline{\int_A} f\,\alpha\mu=s=\sup_\Delta s_\Delta\qquad \text{(Lower Integral)}$$

Theorem 4

Let g be a bounded function of a μ-measurable set A into R, under the same conditions of definition 3 and using the notations

$$\overline{\int} f=\overline{\int_A} f\alpha\mu;\quad \underline{\int} f=\underline{\int_A} f\alpha\mu$$

the next propositions are derived.

(1) If Δ, Δ' are partitions of $A\in H$, belonging to H; then $s_\Delta\leqq S_{\Delta'}$.

(2) $\underline{\int} f\leqq\overline{\int} f$

(3) $\overline{\int} f=-\underline{\int}(-f)$; $\underline{\int} f=-\overline{\int}(-f)$

(4) For any $x\in A$, $f(x)\leqq g(x)$; then

$\overline{\int} f\leqq\overline{\int} g$; $\underline{\int} f\leqq\underline{\int} g$

(5) $\underline{\int} f+\underline{\int} g\leqq\underline{\int}(f+g)\leqq\overline{\int}(f+g)\leqq\overline{\int} f+\overline{\int} g$

(6) $\left|\overline{\int} f\right|$. $\left|\underline{\int} f\right|\leqq\overline{\int}|f|$

(7) It is not always true that

$$\left|\underline{\int} f\right| \leqq \underline{\int} |f|$$

Proof

(1) $\Delta=\{A_1,\cdots, A_n\}$; $\Delta'=\{B_1,\cdots, B_m\}$; $\Delta''=\{A_1\cap B_1,\cdots, A_1\cap B_m,\cdots, A_n\cap B_1,\cdots, A_n\cap B_m\}$

It is easily proved that

$$s_\Delta \leqq s_{\Delta''} \leqq S_{\Delta''} \leqq S_{\Delta'} \qquad \therefore \quad s_\Delta \leqq S_{\Delta'}$$

(2) (1) $\Longrightarrow \sup\limits_{\Delta} s_\Delta \leqq S_{\Delta'}$

$$\therefore \quad \underline{\int} f=\sup_{\Delta} s_\Delta \leqq \inf_{\Delta'} S_{\Delta'}=\overline{\int} f$$

(3) $S(\Delta, f)=\Sigma \sup f(A_i)\cdot\mu(A_i)$

$s(\Delta, -f)=\Sigma \inf(-f(A_i))\cdot\mu(Ai)$

$\sup f(A_i)=-\inf(-f(A_i))$

$\therefore \quad S(\Delta, f)=-s(\Delta, -f)$

$$\therefore \quad \overline{\int} f=\inf_{\Delta} S(\Delta, f)=\inf_{\Delta}(-s(\Delta, -f))=-\sup_{\Delta} s(\Delta, -f)=-\underline{\int}(-f)$$

Hence the first equation is proved.

$s(\Delta, f)=-S(\Delta, -f)$

$$\underline{\int} f=\sup_{\Delta} s(\Delta, f)=\sup_{\Delta}(-S(\Delta, -f))=-\inf_{\Delta} S(\Delta, -f)=-\overline{\int}(-f)$$

(4) $S(\Delta, f)\leqq S(\Delta, g) \Longrightarrow \inf\limits_{\Delta} S(\Delta, f)\leqq S(\Delta, g)$

$$\therefore \quad \overline{\int} f=\inf_{\Delta} S(\Delta, f)\leqq \inf_{\Delta} S(\Delta, g)=\overline{\int} g$$

$$-g\leqq -f \Longrightarrow \overline{\int}(-g)\leqq \overline{\int}(-f)$$

$$\therefore \quad \underline{\int} f=-\overline{\int}(-f)\leqq -\overline{\int}(-g)=\underline{\int} g$$

(5) $\sup(f+g)(A_i)\leqq \sup f(A_i)+\sup g(A_i)$

$\therefore \quad S(f+g, \Delta)\leqq S(f, \Delta)+S(g, \Delta)$

It is derived that

$\inf\limits_{\Delta} S(f+g, \Delta)\leqq \inf\limits_{\Delta} S(f, \Delta)+\inf\limits_{\Delta} S(g, \Delta)$

$$\therefore \quad \overline{\int}(f+g)\leqq \overline{\int} f+\overline{\int} g$$

$$\overline{\int}(-f-g)\leqq \overline{\int}(-f)+\overline{\int}(-g)$$

$$\therefore \quad \underline{\int} f+\underline{\int} g=-\overline{\int}(-f)-\overline{\int}(-g)\leqq -\overline{\int}(-f-g)=\underline{\int}(f+g)$$

(6) $-|f|\leqq f\leqq |f|$

$$\therefore \quad -\overline{\int}|f|=\underline{\int}-|f|\leqq \underline{\int} f\leqq \overline{\int} f\leqq \overline{\int}|f|$$

$\therefore\ \left|\overline{\int} f\right|,\ \left|\underline{\int} f\right| \leqq \overline{\int}|f|$

(7) $A=[0,1]$; $H=\{A,\phi\}$; $\mu(A)=1$; $f(0)=-1$; $f(x)=0$ for $x\neq 0$

$\underline{\int} f=-1\cdot\mu(A)=-1$; $\underline{\int}|f|=0\cdot\mu(A)=0$

$\therefore\ \left|\underline{\int} f\right|=1>0=\underline{\int}|f|$

In this case, (7) is not true.

4. Double Integral

Theorem 5

Let H be a H-Field and $A\in H$, K be a H-Field and and $B\in K$, then $H\times K=\{M\times N;\ M\in H,\ N\in K\}$ is a H-Field and $A\times B\in H\times K$

Proof

(1) $\phi=M\times\phi=\phi\times N\in H\times K$

(2) $M\times N,\ S\times T\in H\times K$

$\Longrightarrow (M\times N)\cap(S\times T)=(M\cap S)\times(N\cap T)$

$\in H\times Q$ (Fig. 2)

(3) $M\times N,\ S\times T\in H\times K$

$M\times N\sim(S\times T)=(M\sim S)\times(N\sim T)$

$\cup(M\sim S)\times(N\cap T)\cup(M\cap S)\times(N\sim T)$

There is a partition $\Delta=\{C_i;\ i=1,\cdots,n\}$ of $M\sim S$ belonging to H and a partition $\Delta'=\{D_j;\ j=1,\cdots,m\}$ of $N\sim T$ belonging to K.

$\therefore\ (M\sim S)\times(N\sim T)=\bigcup_i\bigcup_j C_i\times D_j$

$(M\sim S)\times N\cap T=\bigcup_i C_i\times(N\cap T)$

$(M\cap S)\times(N\sim T)=\bigcup_j (M\cap S)\times D_j$

$\therefore\ M\times N\sim(S\times T)$

$=(\bigcup_i\bigcup_j C_i\times D_j)\cup(\bigcup_i C_i\times(N\cap T))$

$\cup(\bigcup_j(M\cap S)\times D_j)$

$\therefore\ \{C_i\times D_j,\ (M\cap S)\times D_j,\ C_i\times(N\cap T);$

$i=1,\cdots,n;\ j=1,\cdots m\}$

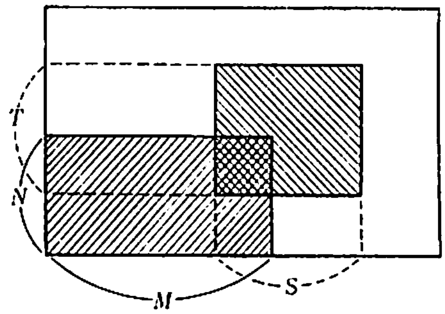
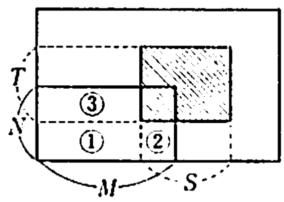


Fig. 2 $(M\times N)\cap(S\times T)$
$=(M\cap S)\times(N\cap T)$



①$=(M\sim S)\times(N\sim T)$
②$=(M\cap S)\times(N\sim T)$
③$=(M\sim S)\times(N\cap T)$
Fig. 3 $M\times N\sim(S\times T)$

is a partition of $M\times N\sim(S\times T)$ belonging to $H\times K$. (Fig. 3) From (1)(2)(3) $H\times K$ is a H-Field and $A\in H,\ B\in K\Longrightarrow A\times B\in H\times K$.

Definition 4

Let H be a H-Field and $A\in H$, K be a H-Field and $B\in K$, then

$$H\times K=\{M\times N;\ M\in H,\ N\in K\}$$

is a H-Field and $A\times B\in H\times K$ from Theorem 5.

$$H\times K \xrightarrow{\kappa} R\sim R^{-}$$

Let κ be *a* measure on $H\times K$,

$$\Delta=\{A_1,\cdots, A_n\}$$

be *a* partition of A belonging to H.

$$\Delta'=\{B_1,\cdots, B_m\}$$

be *a* partition of B belonging to K, then

$$\Delta''=\Delta\times\Delta'=\{A_1\times B_1,\cdots, A_1\times B_m,\cdots, A_n\times B_1\cdots, A_n\times B_m\}$$

is *a* partition of $A\times B$ belonging to $H\times K$.

$$A\times B \xrightarrow[\text{bounded}]{f} R$$

$$S_{\Delta''}=S(\Delta'',f)=\sum_{i=1}^{n}\sum_{j=1}^{m}\sup f(A_i\times B_j)\,\kappa(A_i\times B_j),\ \text{(Sum in Excess)}$$

$$s_{\Delta''}=S(\Delta'',f)=\sum_{i=1}^{n}\sum_{j=1}^{m}\inf f(A_i\times B_j)\,\kappa(A_i\times B_j),\ \text{(Sum in Shortage)}$$

$$\overline{\int}_{A\times B} f d\kappa=\inf_{\Delta''} S\Delta'',\ \text{(Upper Integral)}$$

$$\underline{\int}_{A\times B} f d\kappa=\sup_{\Delta''} s_{\Delta''},\ \text{(Lower Integral)}$$

If μ, ν are defined by the next equations

$$\mu(M)=\kappa(M\times B)\ \text{for each}\ M\in H$$
$$\nu(N)=\kappa(A\times N)\ \text{for each}\ N\in K$$

then it is easily proved that

μ is *a* measure on H,

ν is *a* measure on K.

(i) $\{\mu, \nu\}$ is called to be *independent* iff for each $M\in H$, $N\in K$;

$$\kappa(M\times N)=\mu(M)\times\nu(N)$$

(ii) $\{\mu, \nu\}$ in said to be *dependent* iff there are $M\in H$, $N\in K$ such that

$$\kappa(M\times N)\neq\mu(M)\times\nu(N)$$

Theorem 6

Under the conditions of definition 4, the next propositions are satisfied

(1) $A \xrightarrow[\text{bounded}]{f_1} R$, $\quad B \xrightarrow[\text{bounded}]{f_2} R$

$$\begin{cases} g_1(x, y)=f_1(x)\ \text{for each}\ y\in B \\ g_2(x, y)=f_2(y)\ \text{for each}\ x\in A \end{cases}$$

then g_1, g_2 are bounded functions of $A\times B$ into R, and

$$\overline{\int}(g_1+g_2)d\kappa=\overline{\int} f_1 d\mu+\overline{\int} f_2 d\nu$$

$$\underline{\int}(g_1+g_2)d\kappa=\underline{\int} f_1 d\mu+\underline{\int} f_2 d\nu$$

(2) If $\{\mu, \nu\}$ is independent then

$$\underline{\int} f d\kappa \leqq \begin{cases} \underline{\int}\left[\underline{\int} f(*,\ y)\, d\mu\right] d\nu: & ① \\ \underline{\int}\left[\underline{\int} f(x,\ *)\, d\nu\right] d\mu: & ② \end{cases}$$

$$① \leqq \left\{ \begin{array}{l} \underline{\int}\left[\overline{\int} f(*,\ y) d\mu\right] d\nu \\ \overline{\int}\left[\underline{\int} f(*,\ y) d\mu\right] d\nu \end{array} \right\} \leqq \overline{\int}\left[\overline{\int} f(*,\ y)\, d\mu\right] d\nu: \quad ③$$

$$② \leqq \left\{ \begin{array}{l} \underline{\int}\left[\overline{\int} f(x,\ *) d\nu\right] d\mu \\ \overline{\int}\left[\underline{\int} f(x,\ *) d\nu\right] d\mu \end{array} \right\} \leqq \overline{\int}\left[\overline{\int} f(x,\ *) d\nu\right] d\mu: \quad ④$$

$$\left.\begin{array}{l} ③ \\ ④ \end{array}\right\} \leqq \overline{\int} f d\kappa$$

Proof of (1)

$$\begin{aligned} S(\Delta'',\ g_1+g_2) &= \sum_i \sum_j \sup(g_1+g_2)\ (A_i \times B_j)\ \kappa\ (A_i \times B_j) \\ &= \sum_i \left[\sum_j \sup g_1(A_i \times B_j)\ \kappa\ (A_i \times B_j) \right] \\ &\quad + \sum_j \left[\sum_i \sup g_2(A_i \times B_j)\ \kappa\ (A_i \times B_j) \right] \\ &= \sum_i \sup f_1(A_i)\ \mu\ (A_i) + \sum_j \sup f_2(B_i)\ \nu\ (B_j) \\ &= S(\Delta,\ f_1) + S(\Delta',\ f_2) \end{aligned}$$

It is derived that

$$\overline{\int} (g_1+g_2)\, d\kappa = \overline{\int} f_1 d\mu + \overline{\int} f_2 d\nu$$

using $-g_i$ and $-f_i$, the next equation of (1) follows.

(2) is easily proved.

Acknowledgemnt

Professor Yasusi Hattori in Kyushu University published the book under the title "An Outline of Analysis (Kaiseki Tsuron)" in 1966. The auther of this paper used the book as the text in the lecture of Tokyo University in 1980, and examined the content of the book. In the book Professor Hattori took the view of "H-ring" as the base of Riemann Integral. We call H-ring as Hattori-Field or H-Field.

Without Prof. Hattori's book, this paper will not be written.

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