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A Combinatorial Problem on the Group $Z_3 \times Z_3 \times Z_3$

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Abstract

Let G_k be the direct product of k copies of Z_3 and let $s_3(G_k)$ be the smallest integer s such that, for any sequence of elements of G_k of length s, there is a subsequence of length three or less which sums to the identity. The value of $s_3(G_3)$ and a lower bound of $s_3(G_k)$ for each $k>3$ are given.

1. Introduction

Let G be a finite additive Abelian group. The Davenport constant $s(G)$ is the smallest positive integer s such that, for any sequence $g_1, \cdots, g_s$ of elements of G, there is a nonempty subsequence whose sum equals the identity.

Concerning the generalization of J.E. OLSON's results on this constant, Brian Peterson defined a generalized version of it, $s_r(G)$, as the smallest integer s such that, for any sequence as above, there is a non-empty subsequence of length r or less whose sum is the identity of G, if one exists.

For $G=Z_p \times Z_p \times \cdots \times Z_p$ (the direct product of k copies), where p is a prime and k is a positive integer, he proved that

$$s_p(G) \geqq (2^k-1)(p-1)+1, \qquad (1)$$

and asked when equality holds here. If $k=2$, equality holds by OLSON's result. It is shown in this note that the equality does not hold for $p=3$, $k \geqq 3$.

2. Notations and Results

Let $G_k=Z_3 \times Z_3 \times \cdots \times Z_3$ (k copies). We say that a sequence of elements of G_k has property $P3$ if the sum of every non-empty subsequence of length three or less is not equal to the identity of G_k. Then, $s_3(G_k)$ is the smallest s such that no sequence of length s has property $P3$. We have:

Theorem

$$s_3(G_3)=17, \qquad (2)$$

$$s_3(G_k) \geqq 2^{k+1}+2^{k-2}-1 \quad (k \geqq 3). \qquad (3)$$

3. Proof of the Theorem

Let M be one of the longest sequences of different elements of G_3 with property

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$P3$. Since the order of every element of G_3 which is not the identity 000 is three, the sequence consists of two copies of all elements of M also has property $P3$, so that

$$s_3(G_3)=2|M|+1,$$

where $|M|$ is the number of elements of M.

Regarding G_3 as a vector space over the finite field F_3, it is spanned by M because of maximality of M so that we can find three linearly independent vectors of M. Let f be the linear transformation of G_3 which maps these vectors to 100, 010, and 001, respectively. Then, $f(M)$ has the same property as M has. Therefore we can assume that M contains these three elements. By property $P3$, M does not contain 200, 020, 002, 022, 202 and 220 as well as 000. Besides three elements, 100, 010 and 001, only 17 elements out of 27 elements of G_3 have possibility of belonging to M.

We now construct a graph H which has these 17 elements as its vertices of which two are adjacent if and only if their sum is 000, 200, 020 or 002. H is illustrated in Figure 1. Two sets of vertices of H are called of the same type if they are transformable each other by a sequence of reflections.

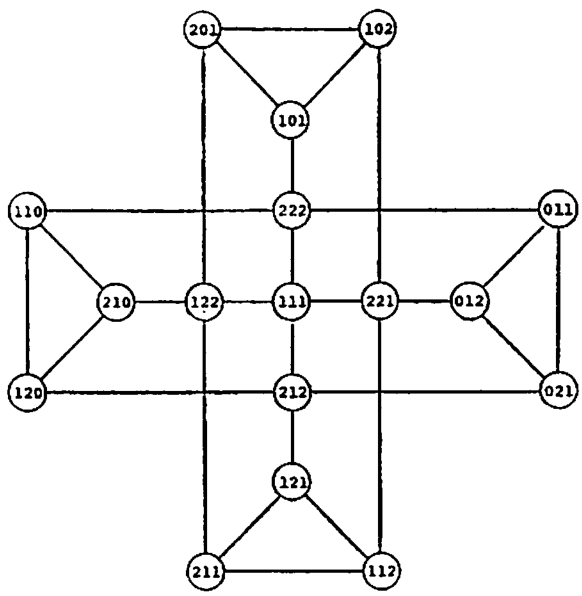

Fig. 1

A set of three vertices of H which sum to 000 must be of the same type as one of the following:

110, 101, 122; 110, 102, 121; 101, 111, 121; 110, 011, 212; 110, 111, 112. (4)

In order to find a set

$$M'=M-\{100,\ 010,\ 001\},$$

all we have to do is to find a set of vertices of H that have the largest number of elements of which no two are adjacent and no three are of the same type as one of (4). It is easy to find that M' is of the same type as one of the following:

101, 210, 111, 112, 021; 110, 122, 121, 221, 011; 110, 122, 212, 012, 102,

from which we have $|M'|=5$. Therefore $|M|=8$, which establishes (2).

If a set N of elements of G_k has property $P3$, we can construct a subset of G_{k+1} of size $2|N|+1$ that has property $P3$ consists of elements, $00\ldots01$ and ones obtained from elements of N by adding 0 or 1 to the right of them. Let M_k denote a subset of G_k which corresponds to M of G_3. Then, we have $|M_{k+1}|\geqq 2|M_k|+1$, from which we have

$$s_3(G_{k+1})\geqq 2s_3(G_k)+1. \qquad (5)$$

From (2) and (5), we get (3) easily.

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