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Geometric Proofs of Cauchy's and Taylor's Theorems

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Abstract

In this paper, we assume Rolle's Theorem as a starting point (Fig. 1) and prove Lagrange's Mean Value Theorem by the method of the equating function F (Fig. 2).

About the proof of Cauchy's theorem, we utilize the inverse function g^{-1} of g (Fig. 3).

We prove Tayloror's Theorem by the method of the approximating function g and the equating function F. (Fig. 4, Fig. 5)

Rolle's Theorem

Let f be continuons on $[a, b]$ and differentiable on $(a, [b)$, and suppose that $f(a)=f(b)$. Then there exists a point $c \in (a, b)$ such that $f'(c)=0$.

Rolle's Theorem is presented as Figure 1.

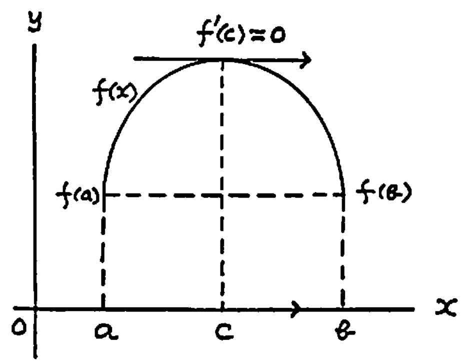

Fig. 1 Rolle's Theorem

Lagrange's Mean Value Theorem.

If f is continuons on $[a, b]$ and differentiable on (a, b) then there exists a point $c \in (a, b)$ such that

$$f(b)=f(a)+f'(c)(b-a)$$

Proof

Let us define the equating function $F(x)$ as follows

$$F(x)=f(x)-m(x-b) \ ; \ m=\frac{f(b)-f(a)}{b-a}$$

then F is continuons on $[a, b]$, and differentiable on (a, b).

$$F(a)=f(b)=F(b)$$

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$F'(x)=f'(x)-m$

By Rolle's Theorem

$o=F'(c)=f'(c)-m$

$=f'(c)-\dfrac{f(b)-f(a)}{b-a}$

$\therefore\ f(b)-(a)=f'(c)(b-a)$

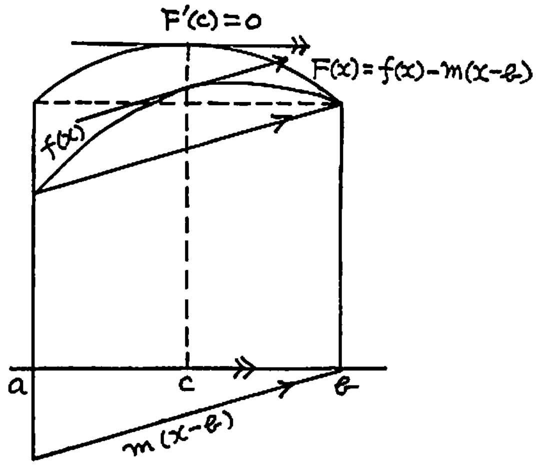


Fig. 2 Lagrange's Mean Value Theorem

Cauchy's Theorem

If f and g are continuons on $[a, b]$ and differentiable on (a, b), and for eaeh $x\in(a, b)$, $g'(x)>0$ [or $g'(x)<0$], then there exists a point $c\in(a, b)$ such that

$$\frac{f(b)-f(a)}{g(b)-g(a)}=\frac{f'(c)}{g'(c)}$$

Proof

we will prove the case, for each $x\in(a, b)$, $g'(x)>0$.

Let $x, x'\in[a, b]$, $x<x'$, then $[x, x']\subseteqq[a, b]$ and $(x, x')\subseteqq(a, b)$. By the assumption, g is continuons on $[x, x']$, and g is differentiable on (x, x').

By Lagrauge's Mean Value Theorem, there exists a point $e\in(x, x')$ such that

$$f(x')-f(x)=f'(e)(x'-x)>0$$

That is g is strictey circreasing function on $[a, b]$.

Let $\alpha=g(a)<g(b)=\beta$

Then there is the inverse function g^{-1} of g, that is g^{-1} on $[\alpha, \beta]$.

Let $t=g(x)$, $x=g^{-1}(t)$, $y=f(x)$

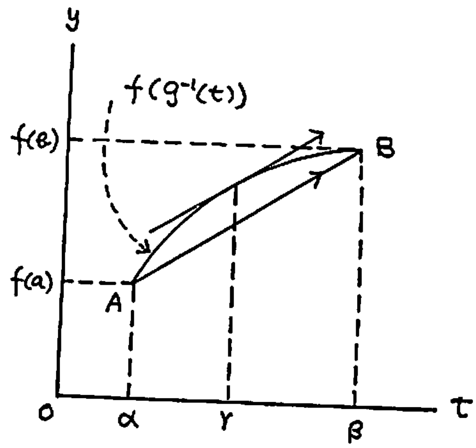
$a=g^{-1}(\alpha)$, $e=g^{-1}(\beta)$

$$t\xrightarrow{g^{-1}}x\xrightarrow{f}y,\qquad t\xrightarrow[f(g^{-1})]{}y$$

$f(g^{-1}(t))$ is continuous on $[\alpha, \beta]$ and differentiable on (α, β).

$$\frac{df(g^{-1}(t))}{dt}=\frac{df(x)}{dx}\cdot\frac{dg^{-1}(t)}{dt}$$

$$=\frac{df(x)}{dx}\Big/\frac{dg(x)}{dx}$$



$A=(g(a), f(a))=(\alpha, f(a))$

$B=(g(b), f(b))=(\beta, f(b))$

Tangent of AB

$=\dfrac{f(b)-f(a)}{\beta-\alpha}=\dfrac{f(b)-f(a)}{g(b)-g(a)}$

Fig. 3 Cauchy's Theorem

By Lagrange's Mean Value Sheorem, there exists a point $\gamma \in (\alpha, \beta)$ such that $c=g^{-1}(\gamma)$.

$$\frac{f(g^{-1}(\beta))-f(g^{-1}(\alpha))}{\beta-\alpha}=\frac{df(g^{-1}(\gamma))}{dt}=\frac{df(c)}{dt}=\frac{df(c)}{dx}\Big/\frac{dg(c)}{dx}$$

$$\therefore \frac{f(b)-f(a)}{g(b)-g(g)}=\frac{f'(c)}{g'(c)}$$

Taylor's Theorem I

If f and its derivative f' are continuons on $[a, b]$ and f' is differentiable on (a, b), then there exists a porut $c \in (a, b)$ such that

$$f(b)=f(a)+f'(a)(b-a)+\frac{1}{2}f''(c)(b-a)^2$$

Proof

We will define the approximating function $g(x)$ of $f(b)$ as follows

$$g(x)=f(x)+f'(x)(b-x)$$

The approximating function $g(x)$ of $f(b)$ is pictured in Fig. 4. Next, we define the equating function F as follows

$$F(x)=g(x)+\frac{(b-x)^2}{(b-a)^2}[f(b)-g(a)]$$

It follows that g is differentiable on (a, b), and

$$g'(x)=f'(x)-f'(x)+f''(x)(b-x)$$
$$=f''(x)(b-x)$$

F is continuons on $[a, b]$ and differentiable on (a, b) and

$$F(a)=f(b)=F(b)$$

$$F'(x)=g'(x)-\frac{2(b-x)}{(b-a)^2}[f(b)-g(a)]$$

$$=f''(x)(b-x)]$$
$$-\frac{2(b-x)}{(b-a)^2}[f(b)-g(a)]$$

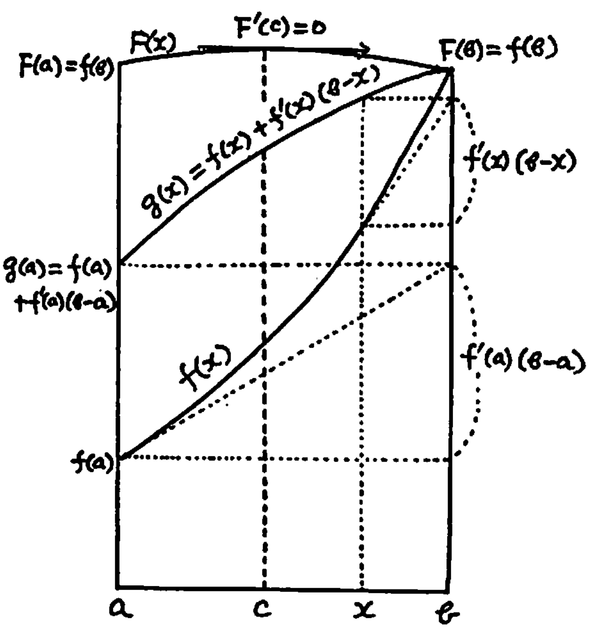


Fig. 4 The approximating function $g(x)$ *of* $f(b)$ and the equating function F.

By Rolle's Theorem, there exists $c \in (a, b)$ such that

$$0=F'(c)=f''(c)\ (b-c)-\frac{2(b-c)}{(b-a)^2}[f(b)-g(a)]$$

$$\therefore\ \ f(b)-g(a)=\frac{1}{2}f''(c)(b-a)^2$$

Example

$$f(x)=1+x+x^2,\quad x \in [0, 1]$$

The approximating function $g(x)$ *of* $f(1)$ is defined as follows

$$g(x)=1+x+x^2+(1+2x)(1-x)$$
$$=2+2x-x^2$$
$$g(0)=2,\ g\left(\frac{1}{2}\right)=2\frac{3}{4},$$
$$g(1)=3=f(1)$$

The equating function F is constant as follows

$$F(x)=g(x)+(1-x)^2[f(1)-g(0)]$$
$$=2+2x-x^2+(1-x)^2=3$$

These functions are pictured in Fig. 5

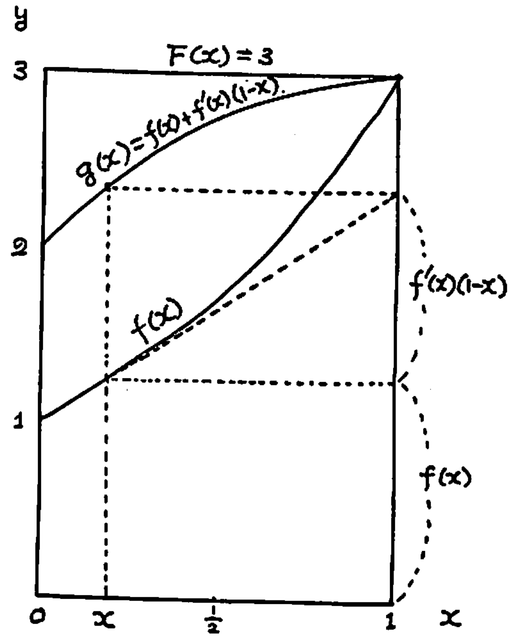


Fig. 5 f, g and F in Example

Taylor's Theorem II

If f and its derivatives f', $f,''$..., $f^{(n)}$ are continuous on $[a, b]$ and $f^{(n)}$ i differentiable on (a, b) then there exists a point $c\in(a, b)$ such that

$$f(b)=f(a)+\frac{1}{1!}f'(a)(b-a)+\frac{1}{2!}f''(a)(b-a)^2$$
$$+\cdots+\frac{1}{n!}f^{(n)}(a)(b-a)^n$$
$$+\frac{1}{(n+1)}f^{(n+1)}(c)(b-a)^{n+1}$$

Proof

The approximating function $g(x)$ of $f(b)$ is defined as follows

$$g(x)=f(x)+f(x)\ (b-x)+\frac{1}{2!}f''(x)(b-x)^2$$
$$+\cdots+\frac{1}{n!}f^{(n)}(x)(b-x)^n$$

From the assumption, g is differentiable on (a, b) and

$$g'(x)=f'(x)-f'(x)+f''(x)(b-x)-f''(x)(b-x)$$
$$+\cdots-\frac{1}{(n-1)!}f^{(n)}(x)(b-x)^{n-1}$$
$$+\frac{1}{n!}f^{(n+1)}(x)(b-x)^n$$
$$=\frac{1}{n!}f^{(n+1)}(x)(b-x)^n$$

The equating function $F(x)$ is defined as follows

$$F(x)=g(x)+\frac{(b-x)^{n+1}}{(b-a)^{n+1}}[f(b)-g(a)]$$

From the assumption, F is continuons on $[a, b]$ and differentiable on (a, b) and

$$F(a)=f(b)=F(b)$$

$$F'(x)=g'(x)-\frac{(n+1)(b-x)^n}{(b-a)^{n+1}}[f(b)-g(a)]$$

$$=\frac{1}{n!}f^{(n+1)}(x)(b-x)^n-\frac{(n+1)(b-x)^n}{(b-a)^{n+1}}[f(b)-g(a)]$$

By Rolle's Theorem, there exists a point $c \in (a, b)$ such that

$$0=F'(c)=\frac{1}{n!}f^{(n+1)}(c)(b-c)^n-\frac{(n+1)(b-c)^n}{(b-a)^{n+1}}[f(b)-g(a)]$$

$$\therefore\ f(b)-g(a)=\frac{1}{(n+1)!}f^{(n+1)}(c)(b-a)^{n+1}$$

Hence, the theorem is proved

Taylor's Theorem III

If f and its n derivatives $f', f'', \cdots, f^{(n)}$ are continuons on $[a, b]$ and $f^{(n)}$ is differentiable on (a, b) then there exists a point $c \in (a, b)$ such that for $m>0$,

$$f(b)=f(a)+\frac{1}{1!}f'(a)(b-a)+\frac{1}{2!}f''(a)(b-a)^2$$

$$+\cdots+\frac{1}{n!}f^{(n)}(a)(b-a)^n$$

$$+\frac{1}{n!m}f^{(n+1)}(c)(b-a)^m(b-c)^{n+1-m}$$

Proof

In the proof of Taylor's Theorem II, if we use the equating function $F(x)$, for positive m

$$F(x)=g(x)+\frac{(b-x)^m}{(b-a)^m}[f(b)-g(a)]$$

where $g(x)$ is the approximating function of $f(b)$.

$$F(a)=f(b)=F(b)$$

From the assumption of the theorem, F is continuons on $[a, b]$ and differentiable on (a, b) and

$$F'(x)=\frac{1}{n!}f^{(n+1)}(x)(b-x)^n-\frac{m(b-x)^{m-1}}{(b-a)^m}[f(b)-g(a)]$$

By Rolle's Theorem, there exists a point $c \in (a, b)$ such that

$$0=F'(c)=\frac{1}{n!}f^{(n+1)}(c)(b-c)^n-\frac{m(b-c)^{m-1}}{(b-a)^m}[f(b)-g(a)]$$

$$\therefore\ f(b)-g(a)=\frac{1}{n!m}f^{(n)}(c)(b-a)^m(b-c)^{n+1-m}$$

Hence the Taylor's Theorem III is proved

$m=1$, then

$$f(b)-g(a)=\frac{1}{n!}f^{(n)}(c)(b-a)(b-c)^n$$

This is called Cauchy's remainder term.

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