Geometric Proofs of Cauchy's and Taylor's Theorems

田中, 穰二 / TANAKA, George

Geometric Proofs of Cauchy's and Taylor's Theorems

George Tanaka*

Abstract

In this paper, we assume Rolle's Theorem as a starting point (Fig. 1) and prove Lagrange's Mean Value Theorem by the method of the equating function F (Fig. 2).

About the proof of Cauchy's theorem, we utilize the inverse function g^{-1} of g (Fig. 3).

We prove Tayloror's Theorem by the method of the approximating function g and the equating function F. (Fig. 4, Fig. 5)

Rolle's Theorem

Let f be continuons on $[a, b]$ and differentiable on $(a, [b)$, and suppose that $f(a) = f(b)$. Then there exists a point $c \in (a, b)$ such that $f'(c) = 0$.
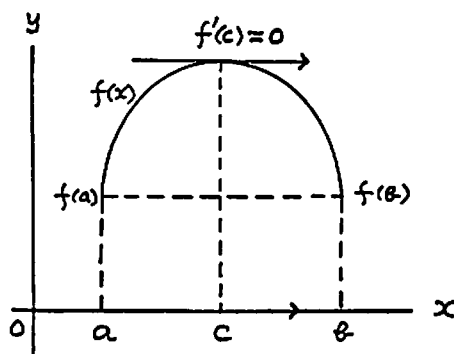
Rolle's Theorem is presented as Figure 1.

Fig. 1 Rolle's Theorem

Lagrange's Mean Value Theorem.

If f is continuons on $[a, b]$ and differentiable on (a, b) then there exists a point $c \in (a, b)$ such that

$$f(b) = f(a) + f'(c)(b-a)$$

Proof

Let us define the equating function $F(x)$ as follows

$$F(x) = f(x) - m(x-b) \ ; \ m = \frac{f(b) - f(a)}{b-a}$$

then F is continuons on $[a, b]$, and differentiable on (a, b).

$$F(a) = f(b) = F(b)$$

---

* Prof. Dr. of Statistics

$$F'(x) = f'(x) - m$$

By Rolle's Theorem

$$o = F'(c) = f'(c) - m$$

$$= f'(c) - \frac{f(b) - f(a)}{b - a}$$

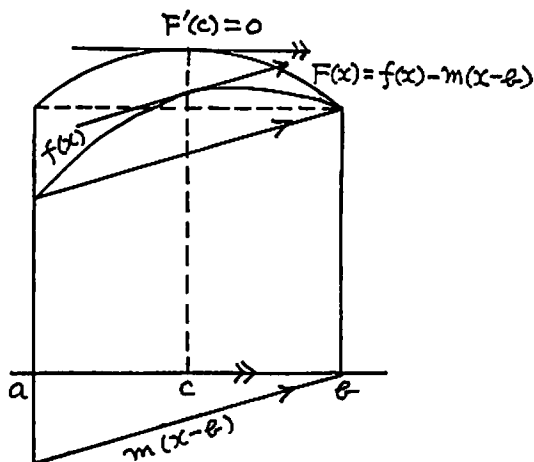$$\therefore \quad f(b) - (a) = f'(c)(b - a)$$

Fig. 2 Lagrange's Mean Value Theorem

Cauchy's Theorem

If f and g are continuons on $[a, b]$ and differentiable on (a, b), and for eaeh $x \in (a, b)$, $g'(x) > 0$ [or $g'(x) < 0$], then there exists a point $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

Proof

we will prove the case, for each $x \in (a, b)$, $g'(x) > 0$.

Let x, $x' \in [a, b]$, $x < x'$, then $[x, x'] \subseteq [a, b]$ and $(x, x') \subseteq (a, b)$. By the assumption, g is continuons on $[x, x']$, and g is differentiable on (x, x').

By Lagrauge's Mean Value Theorem, there exists a point $e \in (x, x')$ such that

$$f(x') - f(x) = f'(e)(x' - x) > 0$$

That is g is strictey circreasing function on $[a, b]$.

Let $\alpha = g(a) < g(b) = \beta$

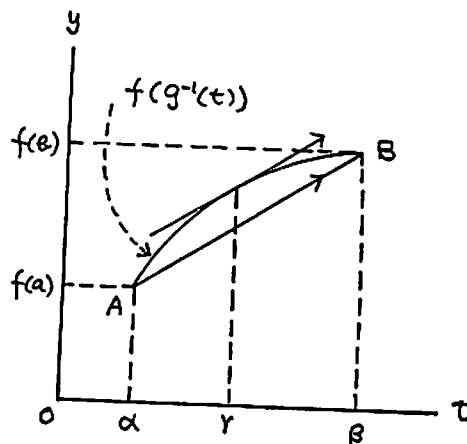Then there is the inverse function g^{-1} of g, that is g^{-1} on $[\alpha, \beta]$.

Let $t = g(x)$, $x = g^{-1}(t)$, $y = f(x)$

$$a = g^{-1}(\alpha), \quad e = g^{-1}(\beta)$$

$$t \xrightarrow{\quad g^{-1} \quad} x \xrightarrow{\quad f \quad} y$$
$$\underbrace{\qquad\qquad}_{f(g^{-1})}$$

$f(g^{-1}(t))$ is continuous on $[\alpha, \beta]$ and differentiable on (α, β).

$$\frac{df(g^{-1}(t))}{dt} = \frac{df(x)}{dx} \cdot \frac{dg^{-1}(t)}{dt}$$

$$= \frac{df(x)}{dx} \bigg/ \frac{dg(x)}{dx}$$

$A = (g(a), f(a)) = (\alpha, f(a))$
$B = (g(b), f(b)) = (\beta, f(b))$

Tangent of AB

$$= \frac{f(b) - f(a)}{\beta - \alpha} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

Fig. 3 Cauchy's Theorem

By Lagrange's Mean Value Sheorem, there exists a point $\gamma \in (\alpha, \beta)$ such that $c = g^{-1}(\gamma)$.

$$\frac{f(g^{-1}(\beta)) - f(g^{-1}(\alpha))}{\beta - \alpha} = \frac{df(g^{-1}(\gamma))}{dt} = \frac{df(c)}{dt} = \frac{df(c)}{dx} \bigg/ \frac{dg(c)}{dx}$$

$$\therefore \frac{f(b) - f(a)}{g(b) - g(g)} = \frac{f'(c)}{g'(c)}$$

Taylor's Theorem I

If f and its derivative f' are continuons on $[a, b]$ and f' is differentiable on (a, b), then there exists a porut $c \in (a, b)$ such that

$$f(b) = f(a) + f'(a)(b-a) + \frac{1}{2} f''(c)(b-a)^2$$

Proof

We will define the approximating function $g(x)$ of $f(b)$ as follows

$$g(x) = f(x) + f'(x)(b - x)$$

The approximating function $g(x)$ of $f(b)$ is pictured in Fig. 4. Next, we define the equating function F as follows

$$F(x) = g(x) + \frac{(b-x)^2}{(b-a)^2}[f(b) - g(a)]$$

It follows that g is differentiable on (a, b), and

$$g'(x) = f'(x) - f'(x) + f''(x)(b-x)$$
$$= f''(x)(b-x)$$

F is continuons on $[a, b]$ and differentiable on (a, b) and

$$F(a) = f(b) = F(b)$$

$$F'(x) = g'(x) - \frac{2(b-x)}{(b-a)^2}[f(b) - g(a)]$$

$$= f''(x)(b-x)]$$

$$-\frac{2(b-x)}{(b-a)^2}[f(b) - g(a)]$$

Fig. 4　The approximating function
$g(x)$ of $f(b)$ and
the equating function F.

By Rolle's Theorem, there exists $c \in (a, b)$ such that

$$0 = F'(c) = f''(c)(b-c) - \frac{2(b-c)}{(b-a)^2}[f(b) - g(a)]$$

$$\therefore f(b) - g(a) = \frac{1}{2} f''(c)(b-a)^2$$

Example

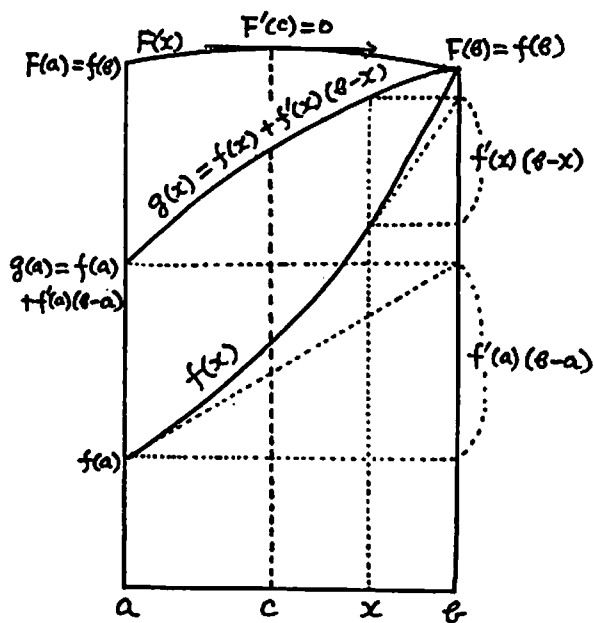$$f(x) = 1 + x + x^2, \quad x \in [0, 1]$$

The approximating function $g(x)$ *of* $f(1)$ is defined as follows

$$g(x) = 1 + x + x^2 + (1 + 2x)(1 - x)$$
$$= 2 + 2x - x^2$$

$$g(0) = 2, \quad g\left(\frac{1}{2}\right) = 2\frac{3}{4},$$

$$g(1) = 3 = f(1)$$

The equating function F is constant as follows

$$F(x) = g(x) + (1 - x)^2 [f(1) - g(0)]$$
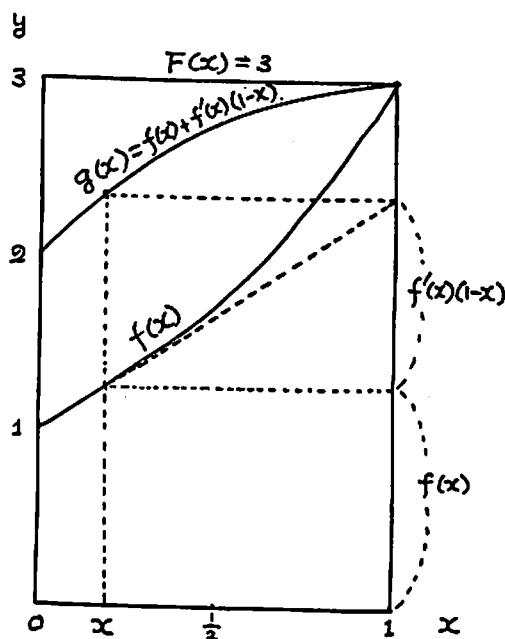$$= 2 + 2x - x^2 + (1 - x)^2 = 3$$

These functions are pictured in Fig. 5

Fig. 5 f, g and F in Example

Taylor's Theorem II

If f and its derivatives f', f''..., $f^{(n)}$ are continuous on $[a, b]$ and $f^{(n)}$ i differentiable on (a, b) then there exists a point $c \in (a, b)$ such that

$$f(b) = f(a) + \frac{1}{1!} f'(a)(b - a) + \frac{1}{2!} f''(a)(b - a)^2$$

$$+ \cdots + \frac{1}{n!} f^{(n)}(a)(b - a)^n$$

$$+ \frac{1}{(n+1)} f^{(n+1)}(c)(b - a)^{n+1}$$

Proof

The approximating function $g(x)$ of $f(b)$ is defined as follows

$$g(x) = f(x) + f(x) (b - x) + \frac{1}{2!} f''(x)(b - x)^2$$

$$+ \cdots + \frac{1}{n!} f^{(n)}(x)(b - x)^n$$

From the assumption, g is differentiable on (a, b) and

$$g'(x) = f'(x) - f'(x) + f''(x)(b - x) - f''(x)(b - x)$$

$$+ \cdots - \frac{1}{(n-1)!} f^{(n)}(x)(b - x)^{n-1}$$

$$+ \frac{1}{n!} f^{(n+1)}(x)(b - x)^n$$

$$= \frac{1}{n!} f^{(n+1)}(x)(b - x)^n$$

The equating function $F(x)$ is defined as follows

$$F(x) = g(x) + \frac{(b-x)^{n+1}}{(b-a)^{n+1}}[f(b) - g(a)]$$

From the assumption, F is continuons on $[a, b]$ and differentiable on (a, b) and
$F(a) = f(b) = F(b)$

$$F'(x) = g'(x) - \frac{(n+1)(b-x)^n}{(b-a)^{n+1}}[f(b) - g(a)]$$

$$= \frac{1}{n!} f^{(n+1)}(x)(b-x)^n - \frac{(n+1)(b-x)^n}{(b-a)^{n+1}}[f(b) - g(a)]$$

By Rolle's Theorem, there exists a point $c \in (a, b)$ such that

$$0 = F'(c) = \frac{1}{n!} f^{(n+1)}(c)(b-c)^n - \frac{(n+1)(b-c)^n}{(b-a)^{n+1}}[f(b) - g(a)]$$

$$\therefore \ f(b) - g(a) = \frac{1}{(n+1)!} f^{(n+1)}(c)(b-a)^{n+1}$$

Hence, the theorem is proved

Taylor's Theorem III

If f and its n derivatives f', f'', $\cdots$, $f^{(n)}$ are continuons on $[a, b]$ and $f^{(n)}$ is differentiable on (a, b) then there exists a point $c \in (a, b)$ such that for $m > 0$,

$$f(b) = f(a) + \frac{1}{1!} f'(a)(b-a) + \frac{1}{2!} f''(a)(b-a)^2$$

$$+ \cdots + \frac{1}{n!} f^{(n)}(a)(b-a)^n$$

$$+ \frac{1}{n!m} f^{(n+1)}(c)(b-a)^m(b-c)^{n+1-m}$$

Proof

In the proof of Taylor's Theorem II, if we use the equating function $F(x)$, for positive m

$$F(x) = g(x) + \frac{(b-x)^m}{(b-a)^m}[f(b) - g(a)]$$

where $g(x)$ is the approximating function of $f(b)$.
$F(a) = f(b) = F(b)$

From the assumption of the theorem, F is continuons on $[a, b]$ and differentiable on (a, b) and

$$F'(x) = \frac{1}{n!} f^{(n+1)}(x)(b-x)^n - \frac{m(b-x)^{m-1}}{(b-a)^m}[f(b) - g(a)]$$

By Rolle's Theorem, there exists a point $c \in (a, b)$ such that

$$0 = F'(c) = \frac{1}{n!} f^{(n+1)}(c)(b-c)^n - \frac{m(b-c)^{m-1}}{(b-a)^m}[f(b) - g(a)]$$

$$\therefore \ f(b) - g(a) = \frac{1}{n!m} f^{(n)}(c)(b-a)^m(b-c)^{n+1-m}$$

Hence the Taylor's Theorem III is proved

$m=1$, then

$$f(b)-g(a)=\frac{1}{n!}f^{(n)}(c)(b-a)(b-c)^n$$

This is called Cauchy's remainder term.

References

Takagi Teiji "Outline of Analysis". (Kaiseki Gairon) Iwanami 1978.

Mimura Yukio "Differential and Integral calculus I, II." (Bisekibun gaku I, II) Iwanami 1973.