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Inference on the Parmenters and Series System Reliability for Two Failure Truncated Weibull Processes

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Abstract

Inference procedures are discussed for the series system reliability for two failure truncated Weibull processes. An lower confideuce limit (LCL) for the series system reliability are obtained.

1. Introduction

In the development of complex systems, the first prototypes produced will contain design and engineering weakeness. To correct these weakeness or deficiencies, these prototypes are subjected to a development testing program. During the testing, weakeness are identified and subsequent redesign are followed out to develop correction for the problem areas. Generally, some modifications are introduced into the system throughout the testing phase. If the modifications introduced into the system during the test are effective, then the system reliability should increase over the testing phase.

Duane[1] proposed to consider the system reliability growth during development testing as a nonhomogeneous Poisson process with intensity $\nu(t)=\lambda\beta t^{\beta-1}$, called a Weibull process.

In the present paper, we shall discuss the problem of inference on the series system reliability when each subsystem is subjected to a independent development testing program. It is assumed that each subsystem produces different nonhomogeneous Poisson processes having intensity,

$$\nu_i(t)=\lambda_i\beta_i t^{\beta_i-1}, \quad i=1,2, \tag{1}$$

with α_i and β_i are unknown positive parameters.

The process will be referred to as failure truncated if it is observed untill the first n failure times, $T_1, T_2, \cdots, T_n$ have occurred.

Suppose each subsystem is in a independent developmental program until a time at which changes in the system cease. If changes cease at the time of the n_i-th failure, t_{n_i}, with respect to each subsystem, and if it is assumed that the intensity, $\nu_i(t_{n_i})$, remains constant thereafter, then the subsequent times between failures of the subsystem will be independent exponential variables with failure rate $\nu_i(t_{n_i})$. If two subsystems

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are independent, the the current series system reliability for mission time $t_0=1$ would be given by,

$$R=\exp[-(\nu_1(t_{n_1})+\nu_2(t_{n_2}))]. \tag{2}$$

In Section 2, we consider the inferences on R, which are closely related to the works of Crow[2] and Bain et al.[3]

2. Inference Results

Suppose data from two Weibull processes are truncated at the n_i-th failure, yielding observed failure times $t_{i1}<t_{i2}<\cdots<t_{in_i}\equiv t_{n_i}$, $i=1,2$.

The MLE's of λ_i, β_i are,

$$\hat{\lambda}_i=n_i/t_{n_i}^{\hat{\beta}_i},\quad \hat{\beta}_i=n_i/\sum_{j=1}^{n_i}\ln(t_{n_i}/t_{ij}),\quad i=1,2. \tag{3}$$

Therefore, MLE's of $\nu_i(t_{n_i})$ are,

$$\hat{\nu}_i(t_{n_i})=\hat{\lambda}_i\hat{\beta}_i t_{n_i}^{\hat{\beta}_i-1}=n_i\hat{\beta}_i/t_{n_i},\quad i=1,2. \tag{4}$$

In Crow,[2] it is shown that for a Weibull process with failure truncated data,

$$\frac{4n^2\nu(t_n)}{\hat{\nu}(t_n)}=Z\cdot S$$

where Z and S are independent Chi squared random variables with respective degrees of freedom $2(n-1)$ and 2n. This result was used to show that,

$$p_r[Z\cdot S/4>\mu]\equiv G(\mu/n)=\int_0^\infty\frac{e^{-x}x^{n-2}}{(n-2)!}\sum_{i=0}^{n}\frac{1}{i!}\left(\frac{\mu}{x}\right)^i\exp\left(-\frac{\mu}{x}\right)dx. \tag{5}$$

But he evaluates this integral by the numerical method.

Next it is shown that above distribution is obtained by another method.

Kotz & Srinivasan[4] have identified the following distributions. The $p.d.f.$ $h(y)$ of the product $Y=X_1\cdot X_2$ of two independent Chi-square variakles with r_1 and r_2 degrees of freedom is,

$$h(y)=\frac{y^{[(r_1/4)+(r_2/4)-1]}K_{(r_1-r_2)/2}(\sqrt{y})}{2^{[(r_1+r_2)/2-1]}\Gamma(r_1/2)\Gamma(r_2/2)},\quad y\geq 0, \tag{6}$$

where $K_m(u)$ is modified Bessel function of the second kind, and $\Gamma(x)$ is the gamma function. In our case, we substitute $r_1=2n$, $r_2=2(n-1)$ into (6) and the $p.d.f.$ of the product $Y=Z\cdot S$ as,

$$h(y)=\frac{y^{n-3/2}K_1(\sqrt{y})}{2^{2n-2}\Gamma(n)\Gamma(n-1)},\quad y\geq 0 \tag{7}$$

The $c.d.f.$ of $h(y)$ is, after some calculations,

$$G(x)=\int_x^\infty h(y)dy=1-\left(\frac{x}{4}\right)^{n-1}\cdot\frac{1}{[\Gamma(n)]^2}+\frac{1}{\Gamma(n)\Gamma(n-1)}\times$$
$$\times\sum_{k=2}^{\infty}\frac{(x/4)^k}{(k-1)(k-2)^2\cdots1^2}\left[\frac{1}{k^2}-\frac{1}{k}\left(\ln\frac{x}{4}+2\nu\right)+\frac{1}{k(k-1)}+\right.$$
$$\left.+\frac{2}{k}\left(\frac{1}{k-2}+\frac{1}{k-3}+\cdots\cdots1\right)\right],\quad n=1,2,\cdots, \tag{8}$$

where $\gamma=0.57722$ (Euler's number).

To estimate the reliability of series system, we substitutc $\hat{\nu}_i(t_{n_i})$, $i=1,2$, into R, and get the MLE as,

$$\hat{R}=\exp[-(\hat{\nu}_1(t_{n_1})+\hat{\nu}_2(t_{n_2})]. \tag{9}$$

The following method to obtain a lower confidence limit (LCL) for $\hat{R}$ of series system has been proposed by Beljaev et al.,[5] To obtain a lower α confidence limit of $\hat{R}$, we need the $p.d.f.$ of $\nu(t_{n_i})/\hat{\nu}(t_{n_i})\equiv Z_i$, $i=1,2$, and it is straightforword to show that the $p.d.f.$ of Z_i is,

$$g_i(z)=\frac{2n_i^{n_i-1/2}z^{n_i-3/2}K_1(2\sqrt{n_i z})}{\Gamma(n_i)\,\Gamma(n_i-1)},\quad z\geq 0,\quad i=1,2. \tag{10}$$

This result is used to obtain,

$$P_r\left\{\frac{\nu(t_{n_1})}{\hat{\nu}(t_{n_1})}+\frac{\nu(t_{n_2})}{\hat{\nu}(t_{n_2})}<A_\alpha(n_1,n_2)\right\}=\alpha \tag{11}$$

where $A_\alpha(n_1, n_2)$ is the constant depending on α, n_1, n_2.
The event,

$$\sum_{i=1}^{2}\frac{\nu(t_{n_i})}{\hat{\nu}(t_{n_i})}<A_\alpha(n_1,n_2) \tag{12}$$

can be regarded as a set of inequalities inposed on the parameters $\nu(t_{n_i})$ the probability of satisfaction of which is equal to α.

If we consider the maximum of the function,

$$\bar{\psi}=\max\{\sum_{i=1}^{2}\nu_i(t_{n_i})\} \tag{13}$$

On the set defined by inequalities (12), then, obviously,

$$\{(\nu_1(t_{n_1}),\ \nu_2(t_{n_2})\,|\,\bar{\psi}=\max\{\sum_{i=1}^{2}\nu_i(t_{n_i})\}\geq\sum_{i=1}^{2}\nu_i(t_{n_i})\}$$

$$\supseteq\{(\nu_1(t_{n_1}),\ \nu_2(t_{n_2})\,|\sum_{i=1}^{2}\frac{\nu(t_{n_i})}{\hat{\nu}(t_{n_i})}<A_\alpha(n_1,n_2)\} \tag{14}$$

From this, we obtain,

$$P_r\{R=\exp[-(\sum_{i=1}^{2}\nu_i(t_{n_i}))]\geq\underline{R}=\exp(-\bar{\psi})\}$$

$$=P_r\{\bar{\psi}\geq\sum_{i=1}^{2}\nu_i(t_{n_i})\}\geq P_r\{\sum_{i=1}^{2}\frac{\nu(t_{n_i})}{\hat{\nu}(t_{n_i})}<A_\alpha(n_1,n_2)\}=\alpha. \tag{15}$$

Thus, the problem of finding the lower confidence limit for $\hat{R}$ is reduced to the problem of finding the maximum of the function (13). The answer is found in explicit form :

$$\bar{\psi}=\max_{1\leq i\leq 2}\ \hat{\nu}_i(t_{n_i})\,A_\alpha(n_1,n_2). \tag{16}$$

Thus, the LCL with confidence coefficient no less than α is,

$$\underline{R}=\exp\{-(\max_{1\leq i\leq 2}\hat{\nu}_i(t_{n_i})\,A_\alpha(n_1,n_2))\}. \tag{17}$$

To calculate these $A_\alpha(n_1, n_2)$ velues, we need the $c.d.f.$ of,

$$T=\sum_{i=1}^{2}\frac{\nu(t_{n_i})}{\hat{\nu}_i(t_{n_i})}=Z_1+Z_2.$$

We consider two method of approximating the $c.d.f.$ of T.

Method 1.[6)]

Z_i is the H-function $r.v.$, and T is the sum of independent H-function $r.v.$.
Therefore approximating $p.d.f.$ can be obtained. We can evaluatc the accuracy of the approximation by the method developed by Posten & Woods.[7)]

Using the first 10 moments of the exact and the approximate distribations, we can evaluate the accuracy of the Laguerre approximation $K(t)$ to the cumulative true distribution $F(t)$ of the sum $T=Z_1+Z_2$ of the $i.r.v$'s with $p.d.f.$'s given by (10), where the Laguerre $p.d.f.$ is based on the first three moments of $f(t)$ which is $p.d.f.$ of $F(t)$. Then the moments of the corresponding transformed density fecnction $p(y)$ which correspond to $k(t)$ and $q(y)$ which correspond to $f(t)$ must be determined, where $y=e^{-t}$.

From a knowledge of these moments, the accuracy of the approximate distribution may be evaluated. First, the $m.g.f.$'s of $g_i(z)$, $i=1,2,$ is calculated and denoted by $M_{z_i}(r)$.

Hence the $m.g.f.$ of $f(t)$ is,

$$M_t(r)=M_{z_1}(r)\cdot M_{z_2}(r)$$

so the moments are,

$$\mu'_{k:f(t)}=\left[\frac{d^k}{dr^k}M_t(r)\right]\Big|_{r=0}. \tag{18}$$

It then follows immediately from (18) that the first three moments of the exact $p.d.f.$ $f(t)$ are, when $n_1=n_2=2$, $\mu'_{0:f(t)}=1$, $\mu'_{1:f(t)}=2$, $\mu'_{2:f(t)}=8$, $\mu'_{3:f(t)}=54$.

The Laguerre polynomial ($p.d.f.$) based on these three moments is,

$$k(t)=\sum_{j=0}^{3}d_j L_j^{(r)}\phi(t),\ \ 0\leq t<\infty,\ \ r=0. \tag{19}$$

where,

$$d_n=\frac{(-1)^n}{n!(1+r)_n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l\frac{(1+r)_n}{(1+r)_l}\mu'_{l:f(t)},\ n=0,1,2,3,$$

$$\phi(t)=\frac{t^r e^{-t}}{\Gamma(r+1)}, \tag{20}$$

and $(n)_m\equiv n(n+1)(n+2)\cdots\cdots(n+m-1)$

$$L_0^{(0)}(t)=1,\ L_1^{(0)}(t)=t-1,\ L_2^{(0)}(t)=t^2-4t+2,\ L_3^{(0)}(t)=t^3-9t^2+18t-6.$$

On evaluating (20) for $n=0,1,2,3$, one obtains,

$$d_0=1,\ d_1=1,\ d_2=0.5,\ d_3=0.3333.$$

Also, from (21) with $r=0$, one has,

$$\phi(t)=e^{-t}$$

Using these results in (19), one obtains the desired Laguerre $p.d.f.$

$$k(t)=[-1+5t-2.5t^2+0.3333t^3]e^{-t} \tag{21}$$

It remains to determine the first ten moments of $p(y)$ and $q(y)$, denoted, respectively, by $\mu'_{k:p(y)}$ and $\mu'_{k:q(y)}$, $k=1,2,\cdots,10$, where $y=e^{-t}$. Specifically, since $t=-\ln y$, the transformed Laguerre density function is,

$$p(y)=\left[-1+5\left(\ln\frac{1}{y}\right)-2.5\left(\ln\frac{1}{y}\right)^2+0.3333\left(\ln\frac{1}{y}\right)^3\right],$$

and the k-th moments is,

$$\mu'_{k:p(y)}=E[y^k]=\int_0^1 y^k\,p(y)\,dy.$$

In pacticular,

$\mu'_{1:p(y)}=$ 0.249878, $\mu'_{2:p(y)}=$ 0.061706, $\mu'_{3:p(y)}=-0.007818$, $\mu'_{4:p(y)}=-0.036801$, $\mu'_{5:p(y)}=-0.049382$, $\mu'_{6:p(y)}=-0.054560$, $\mu'_{7:p(y)}=-0.056151$, $\mu'_{8:p(y)}=-0.055935$, $\mu'_{9:p(y)}=-0.054799$, $\mu'_{10:p(y)}=-0.053201$.

Similary, $f(t)$ is transformed into the densty fanction $q(y)$ whose k-th moment is,

$\mu'_{1:q(y)}=0.307678$, $\mu'_{2:q(y)}=0.162965$, $\mu'_{3:q(y)}=0.103625$, $\mu'_{4:q(y)}=0.072643$, $\mu'_{5:q(y)}=0.054163$, $\mu'_{6:q(y)}=0.042150$, $\mu'_{7:q(y)}=0.033852$, $\mu'_{8:q(y)}=0.027856$, $\mu'_{9:q(y)}=0.023368$, $\mu'_{10:q(y)}=0.019912$.

Finally, to evaluate the error of the approximation, one utilize Goertzel's algorithm[6)]

$$U_k=d_k^*+(4y-2)U_{k+1}-U_{k+2},$$

to determine, respectively, the quantities U_k, $k=10, 9, \cdots, 1$, from which one then determines the error $S=2U_1\sqrt{y-y_2}$ incurred by using the distribation function $P(y)$ to estimate $Q(y)$ for a specified ualue of y, where,

$$d_k^*=(-1)^k\left(\frac{2}{\pi}\right)\sum_{j=1}^{k}\frac{(-4)^j(j+k-1)!}{(k-j)!\,(2j)!}(\mu'_{j:q(y)}-\mu'_{j:p(y)}).$$

The values of d_k^* are given in Table 1. The error in $c.d.f.$ $P(y)$, denoted by $\varepsilon(P(y))$, is evaluated for $y=1.0$, 0.8, 0.7, 0.6, 0.4, 0.3, 0.2, 0.1 and for those values of y correspording to $t=E[T]+k\sigma_T$, $k=0, 1, 2, 3$. Since $y=e^{-t}$ varies inversely with t, $p(y)$ corresponds to the complimentary $d.f.$ $\bar{K}(t)=1-K(t)$, where $t=-\ln y$, $0<y\leq 1$. That is,

$$\varepsilon(P(y))=-\varepsilon(K(t))$$

Thus, the error in $P(y)$ is numerically equal to that in $K(t)$ but opposite in sign. Finally, since $k(t)$ maps into $p(y)$ and $f(t)$ maps into $q(y)$, it follows that $1-(K(t)-\varepsilon K(t))$ is a valid approximation to the exact $d.f.$ $Q(y)$.

From Table 1, the values $A_a(2, 2)$ are obtained.

Table 1. Enaluation of Accuracy of the Lagunerre Approximation of distribution of $T=Z_1+Z_2$

k	d_k^*	t	y	$1-K(t)$	Error: S	$\bar{F}(t)=1-K(t)+\varepsilon K(t)$	$F(t)=1-\bar{F}(t)$
1	0.73592	0	1.0	1.	0.	1.	0.
2	0.11067	0.2231	0.8	1.10024	0.133396	0.966844	0.033156
3	-0.0538738	0.3567	0.7	1.07634	0.126366	0.948974	0.0500261
4	-0.0268706	0.5108	0.6	1.00481	0.287587	0.717223	0.282777
5	0.0166173	0.9163	0.4	0.731845	-0.0194449	0.75129	0.24871
6	-0.0174076	1.2040	0.3	0.544591	0.122806	0.421785	0.578215
7	0.0105522	1.6094	0.2	0.34463	-0.15933	0.50396	0.49604
8	-0.0034582	2	0.135335	0.225558	-0.264553	0.490111	0.509889
9	-0.0436571	2.3026	0.1	0.172164	-0.120987	0.293151	0.706849
10	0.114641	4	0.0183156	0.115989	-0.0818983	0.197887	0.802113
11		6	0.00247875	0.0768257	-0.123367	0.200193	0.799807
12		8	0.000335463	0.037332	-0.0520918	0.0894238	0.910576

Method 2.

Let the $p.d.f.$ of T bc $f(t)$. We obtain after some calculations,

$$f(t)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}e^{rt}\Big[n_1^{n_1-1}\,n_2^{n_2-1}\,r^{-(n_1+n_2)+2}e^{\frac{n_1+n_2}{2r}}\times$$

$$\times W_{-n_1+1,\frac{1}{2}}\left(\frac{n_1}{r}\right)\cdot W_{-n_2+1,\frac{1}{2}}\left(\frac{n_2}{r}\right)\Big]dr,$$

where $W_{k,\mu}(z)$ is Whittaker's function.

This inversion of integral transform can be obtiain by numerical method. But in this paper we don't discuss this mumerical evaluation of this integral tarnsform further.

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