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Abstruct

A mathmatical model permits determing the duration of payoff-optimized burn-in testing program. It is assumed that system failure may be due to either poor components or good components. The payoff is difference between the s-expected profit in the field activities and the total testing cost. The mathmatical tool is the optimal stopping problem of Markov chain. A numerical example illustrates these concepts.

1. Introduction and Assumptions

Today we have very complex systems in our society and the factories. Before the system is used, a burn-in testing is usually considered as effective means of eliminating early failures due to deffective elments in a laboratory environment. Suppose the system consists of J components. It is assumed that system failure may be due to either poor components. or good components. Before testing begin, the system has $i(i=0, \cdots, J)$ poor components. (Kigawa, S. [1])

The following assumptions are made:

(1) The failure rate of the good components is ϕ_o, and the failure rate of the poor components is ϕ $(\phi_o<\phi)$.

(2) When the component fails, the failed component is replaced by the new one. The probability that the new component is good one is $(1-p)$, therefor the probability that the new one is poor one is p.

(3) The testing cost per time is c (yen)/time·

(4) After we stop the burn-in program, untill the system failure in the field activities we acquire the profit r (yen)/time. After the system fails, it is supposed that the system can't be used any more.

The s-expected profit associated with the field activities are tradeoff with the costs of implimenting a burn-in testing program.

A mathmatical model permits determining the duration of optimized burn-in. A numerical example illustrates these concepts.

2. Formulation of the problem (Dynkin etal. [2])

Let the system exist at each instant of time in one of the states formed by a finite or

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demumerable set E (state space). Here we consider the state as the number of poor components, therefor we set $E=\{0, \cdots, J\}$.

The process just described is a birth and death process $Y=\{Y(t); t\epsilon R_+\}$ with state space E and transition function

$P_{ij}(t)=P\{Y(t+s)=j|Y(s)=i\}$,

for any t, $s\geq 0$ and i, $j\epsilon E$.

This process satisfies the following conditions,

(1) $P_{i,i+1}(h)=\lambda_i h+o(h)$ $(h\downarrow 0)$, $0\leq i\leq J-1$,

(2) $P_{i,i-1}(h)=\mu_i h+o(h)$ $(h\downarrow 0)$, $0\leq i\leq J$,

(3) $P_{i,i}(h)=1-(\lambda_i+\mu_i)h+o(h)$ $(h\downarrow 0)$, $0\leq i\leq J$,

(4) $P_{i,j}(0)=\delta_{ij}$,

(5) $\mu_i=i\phi(1-p)$, $\lambda_i=(J-i)\phi_o p$.

where $o(h)$ possibly depends on i. The matrix

$$A=\begin{pmatrix} -\lambda_o & \lambda_o & 0 & 0 & \cdots & 0 & 0 \\ \mu_1 & -(\lambda_1+\mu_1) & \lambda_1 & 0 & \cdots & 0 & 0 \\ 0 & \mu_2 & -(\lambda_2+\mu_2) & \lambda_2 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdots & \mu_J & -\mu_J \end{pmatrix}$$

is the generator of this process.

The underlying Markow chain $X=\{X(n): n\epsilon N\}$ where $N=\{0, 1\cdots\}$, has the trantion probabilities

$$Q(i, i-1)=q_i=\frac{\mu_i}{\mu_i+\lambda_i},$$

$$Q(i, i+1)=p_i=\frac{\lambda_i}{\mu_i+\lambda_i},$$

$Q(i, j)=0$ $(|i-j|\neq 1)$, i, $j\epsilon E$.

The time ξ_i from arrival at the point i to exit from this point is distributed according to the exponential law

$P\{\xi_i\geq t\}=\exp\{-(\lambda_i+\mu_i)t\}$ $(t\geq 0)$.

From now we consider the underlying Markov chain X and $r.v.$ ξ_i. Let us suppose that we observe the path $X(0), X(1), \cdots, X(n)$, and can at any instant n stop the migration system. If at the time of stopping the system is stuated at the point i, we acquire a profit $f(i)$. In our case, $f(i)$ is the s-expected profit untill the first system failure, therefor

$$f(i)=r\int_0^\infty \exp\{-[(J-i)\phi_o+i\phi]t\}dt$$

$$=\frac{r}{(J-i)\phi_o+i\phi}.$$

The costs associatd with burn in testing is c (yen)/time. Clearly, the total payoff at X

(n) is $f(X(n))-c\tau_n$, where τ_n is the time for the n-th failure. The problem is stated as follows:

The underlying Markov chain X with trantion probability $Q(i, j)$ and the total payoff $f(X(n))-c\tau_n$ are given on E.

It is required to:

1) calcuate the variable $v(k)=\sup_n M_k(f(X(n))-c\tau_n)$, where n represent all the possible Markov times and M_k indicates the expectation for the initial position of the system at the point k.

2) find the Markov time n_o for which $M_k(f(X(n_o))-c\tau_{n_o})=v(k)$, Q denote an operator to the formula for a function φ

$Q\varphi(k)=\sum_{\ell} Q(k, \ell)\varphi(\ell)$ (the one-step shift operator).

By analogy with the theory of games, the variable $v(k)$ is called the value of game, and the Markov time n_o is called the optimal stopping time.

Next we will show,

$$v(k)\geq Qv(k)-cM_k\tau_1. \qquad (1)$$

We pick an arbitrary number $\varepsilon>0$ and denote by m the stopping time for which

$M_\ell\{f(X(m))-c\tau_m\}\geq v(\ell)-\varepsilon$, $(\ell\in E)$.

Obcause m is a function of ℓ, $X(1)$, $X(2)$, $\cdot\cdot\cdot$: say $m=t(\ell, X(1), X(2), \quad)$. Now $n=1+t(X(1), X(2), \cdot\cdot\cdot)$; that is, n is the stopping time corresponding to the strategy which waits at $X(0)$, and if $X(1)=\ell$, then uses the stortegy corresponding to m thereafter. Then for any $k\in E$,

$$\begin{aligned} v(k) &\geq M_k\{f(X(n))-c\tau_n\}=M_k f(X(n))-cM_k\tau_n \\ &=\sum_{\ell} Q(k, \ell)M_\ell f(X(m))-cM_k\tau_1-cM_\ell\tau_m \\ &=\sum_{\ell} Q(k, \ell)M_\ell\{f(X(m))-c\tau_m\}-cM_k\tau_1 \\ &\geq\sum_{\ell} Q(k, \ell)\{v(\ell)-\varepsilon\}-cM_k\tau_1 \\ &=Qv(k)-\varepsilon-cM_k\tau_1 \end{aligned}$$

$\therefore v(k)\geq Qv(k)-cM_k\tau_1$.

Since $n=0$ is a possible stopping time, $v(k)=\sup_n M_k(f(X(n))-c\tau_n)\geq M_k f(X(0))=f(k)$ for all k.

To show that v is the minimal such funcion, let g be an another function satisfying (1) and suppose $g\geq f$.

For any stopping time n, since $M_k f(X(n))\leq M_k g(X(n))$,

$M_k\{f(X(n))-c\tau_n\}\leq M_k\{g(X(n))-c\tau_n\}\leq g(k)$,

hence $v(k)=\sup_n M_k\{f(X(n)-c\tau_n\}\leq g(k)$.

Summarizing, we have deduced that the value of the game v is the minimum function satisfying;

$$\begin{cases} v(k) \geq \sum_{\ell} Q(k, l)v(l) - cM_k\tau_1, \\ v(k) \geq f(k), & (k \in E) \\ v(k) \geq 0. \end{cases}$$

In our case these equations become

$$\begin{cases} v(k) \geq p_k v(k+1) + q_k v(k-1) - \dfrac{c}{(J-k)\phi_o p + k\phi(1-p)}, \\ v(k) \geq \dfrac{r}{(J-k)\phi_o + k\phi}, \\ v(k) \geq 0. \end{cases}$$

Therfore we have

$$v(k) = \max\left\{ \frac{r}{(J-k)\phi_o + \phi\lambda},\ p_k v(k+1) + q_k v(k-1) - \frac{c}{(J-k)\phi_o p + k\phi(1-p)} \right\}.$$

3. The Optimal Stopping time

We denote by Γ the set of all state k in which the profit function $f(k)$ is equal to $v(k)$. We call this set the support set. Before giving a numerical example, we give the following main result characterizing the optimal stopping time.

THEOREM. 1

Suppose the state space E is finite. Then the time n_o of the first visit to the support set Γ is an optimal stopping time.

To prove this we shall need the following Lemma

LEMMA. 1.

Let m be the time of first visit to a fixed set Γ of states. If g satisfies (1), *then the function h defined by*

$h(k) = M_k[g(X(m)) - c\tau_m]$, $k \in E$,

also satisfies (1).

(*Proof*).

Let m be the time to first visit to Γ, and let n be the time of first visit to Γ at or after time 1; that is, let

$m = \inf\{\ell \geq 0;\ X(\ell) \in \Gamma\}$,

$n = \inf\{\ell \geq 1;\ X(\ell) \in \Gamma\}$,

If $X(0) \in \Gamma$, then $m = 0 \leq 1 \leq n$; if $X(0) \notin \Gamma$, then $m = \inf\{\ell \geq 1: X(\ell) \in \Gamma\} = n$. Hence, $m \leq n$; and it is clear that both m and n are stopping times.

Since g setisfies (1),

$$h(k) = M_k[g(X(m)) - c\tau_m] \geq M_k[g(X(n)) - c\tau_m] \geq M_k[g(X(n)) - c\tau_n],\ k \in E \qquad (2)$$

On the other hand, if $m = t(X(0), X(1), \cdots)$ for some function t, then $n = 1 + t(X(1), X(2), \cdots)$; hence

$M_k[g(X(n))-c\tau_n|X(1)=j]=M_j[g(X(m))-c\tau_m]=h(j)$

which inplies that

$M_k[g(X(n))-c\tau_n]=\sum_j Q(k, j)h(j).$

This togather with (2) show that $h(k)\geq Qh$ and since $h\geq 0$ obvioulу, h satisfies (1). ■

(*Proof of Theorem 1.*)

Let us examine the average total payoff

$$h(k)=M_k\{f(X(n_o))-c\tau_{n_o}\} \quad (3)$$

which corresponds to the stopping time n_o. It is required to prove that $h=v$. According to the actual definition of the value of the game $h\leq v$. To show the reverse inequality, we will first show that h satisfies (1) and $h\geq f$; then since v is the minimal function satisfying (1) and $v\geq f$, $v\leq h$ as well.

Inasmuch as $X(n_o)\in\Gamma$ while f and v coincide on Γ, the function f may be replaced in eq. (3) by v, and (3) becomes

$$h(k)=M_k[v(X(n_o))-c\tau_{n_o}],\ k\in E. \quad (4)$$

The function v satisfies (1) and by Lemma 1, (2) implies that h also satisfies (1).

Next we show that $h\geq f$, For $k\in\Gamma$, $P_k\{n_o=0\}=1$ and therefore

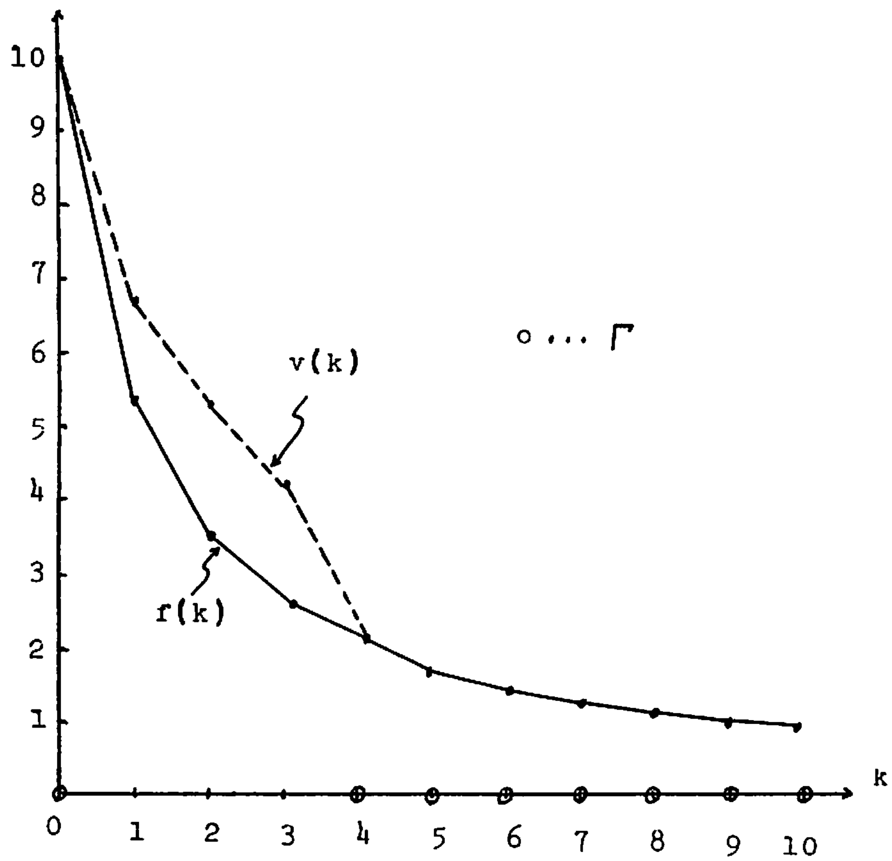

Fig. 1

$h(k)=M_k[f(X(n_o))-c\tau_{n_o}]=f(k).$

Suppose for a moment that for some $k\notin\Gamma$, $h(k)<f(k)$. Since E is finite, there is a state $j\notin\Gamma$ at which the defference $f(k)-h(k)$ is maximized; let $c=f(j)-h(j)$ be this maximum value. By the way c is picked, $h_1(k)=h(k)+c\geq f(k)$ and $h_1(k)$ coincides with $f(k)$ at the point j, and, as the sum of $h(k)$ (satisfying (1)) and the positve constant c, is also satisfies (1). Consequently, $h_1(k)\geq v(k)$ and $f(j)=h_1(j)\geq v(j)$. This means that the point j chosen outside the support set Γ belongs to Γ. The ensuring contradiction reveals the inequality $h(k)<f(k)$ is inadmissible. The optimality of the strategy n_o is thus proved. ■

Next we will find the support set Γ.

$$v(k)=\max\left\{f(k),\ p_k v(k+1)+q_k v(k-1)-\frac{c}{a_k}\right\},$$

where $a_k=(J-k)\phi_o p+k\phi(1-p)$.

When $v(k)=f(k)$, since we have

$$v(k)\geq p_k v(k+1)+q_k v(k-1)-\frac{c}{a_k},$$

we get

$$f(k)\geq p_k f(k+1)+q_k f(k-1)-\frac{c}{a_k},\text{ therefore,}$$

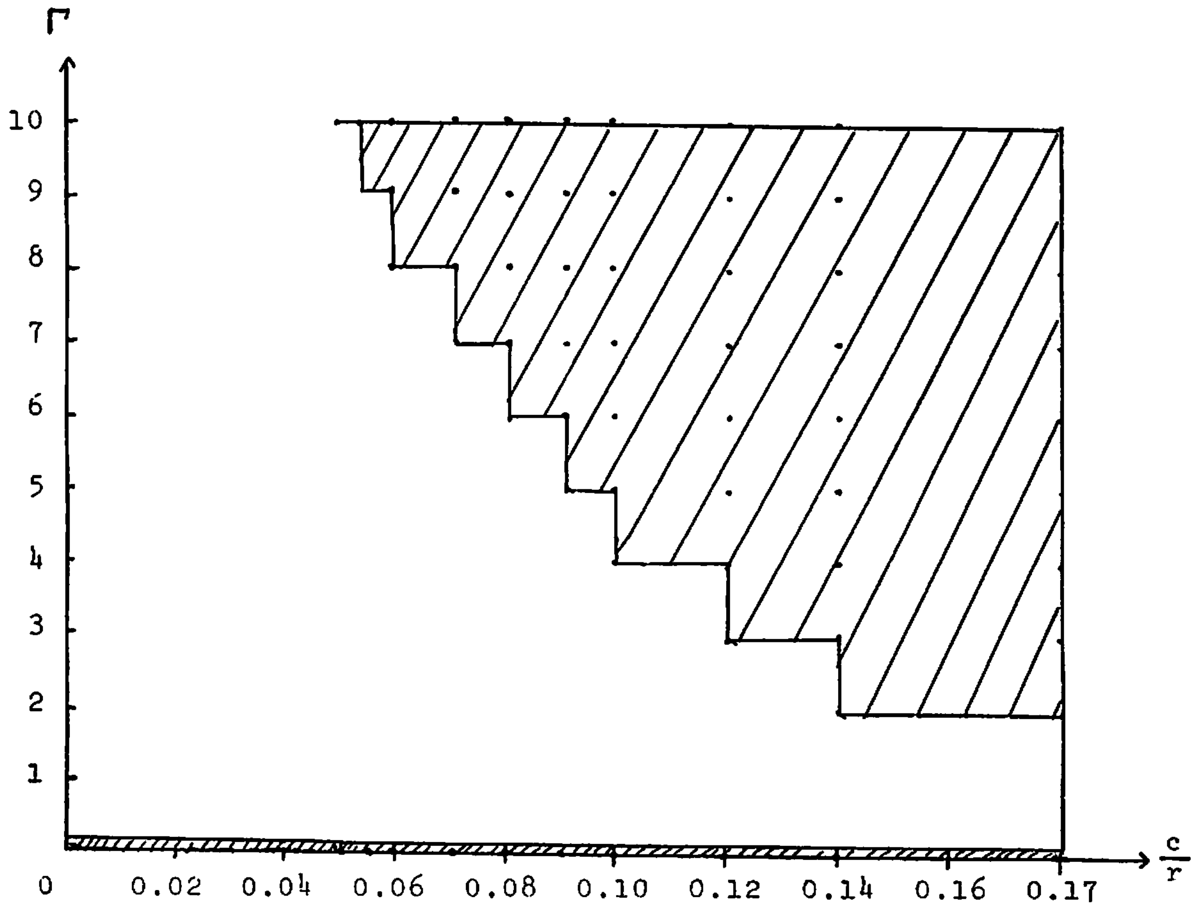

Fig. 2

$$\frac{k\phi(1-p)+(J-k)\phi_o p}{(J-k)\phi_o+k\phi} \geq \frac{(J-k)\phi_o p}{(J-k-1)\phi_o+(k+1)\phi}+\frac{k\phi(1-p)}{(J-k+1)\phi_o+(k-1)\phi}-\frac{c}{r}.$$

[Numerical Example]

$N=10$, $\phi_o=0.1$, $\phi=1$, $p=0.5$, $c/r=0.1$.

$\Gamma=\{0, 4, 5, \cdot\cdot\cdot, 10\}$.

It $k\in\Gamma$, we hsve $v(k)=f(k)=\frac{r}{(J-k)\phi_o+k\phi}$

It $k\notin\Gamma$, we have $v(k)=p_k v(k+1)+q_k v(k-1)-\frac{c}{a_k}$ and from this equation, $v(1)=6.67$, $v(2)=5.14$, $v(3)=4.03$.

Fig. 1 indicates a graph of the function $f(k)$ and $v(k)$ in the above case.

Fig. 2 represents Γ varying c/r frome 0.00 to 0.17.

References

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