On the Superposition of the Two Independent Renewal Processes

KIGAWA, Shun’ichi

(出版者 / Publisher)
法政大学工学部
(雑誌名 / Journal or Publication Title)
法政大学工学部研究集報 / 法政大学工学部研究集報
(巻 / Volume)
22
(開始ページ / Start Page)
221
(終了ページ / End Page)
225
(発行年 / Year)
1986-03
(URL)
https://doi.org/10.15002/00004026

On the Superposition of the Two Independent Renewal Processes

Shun'ichi KIGAWA

Abstract

Cox [1] discussed the superposition of p independent identical renewal processes and derived the asymptotic mean of the time up to the (r)th event in the superposed process. We extend the above result to the independent but not identical renewal processes. Some numerical examples are presented.

1. Introduction

A renewal process can be viewed as a sequence of points $\{{}^1T_n;\ n\geq 0\}$ where $0={}^1T_0\leq{}^1T_1\leq\cdots$. The random variables X_n defined by $X_n={}^1T_n-{}^1T_{n-1}$ for $n\geq 1$ are assumed to be independent and identically distributed. But in this paper the renewal processes in question will be denoted by $\{{}^1T_n;\ n\geq 0\}$ and $\{{}^2T_m;\ m\geq 0\}$ with increments $\{X_n;\ n\geq 1\}$ and $\{Y_m;\ m\geq 1\}$ respectively. By the superposition of the two renewal processes we shall mean the following:

Let $\{{}^1T_n\}_{n=0}^{\infty}$ and $\{{}^2T_m\}_{m=0}^{\infty}$ be realization of the two processes. Viewing these realizations as sets of points, we superpose the two processes simply by taking the uion of the two sets and reordering the indexes to produce a single monotone increasing sequence $\{U_k;\ k\geq 0\}$. The asymptotic mean of U_r will be derived and some numerical examples will be presented. For purpose of illutration, in this paper we refer to the failure process of a device consisting of two independent components in series.

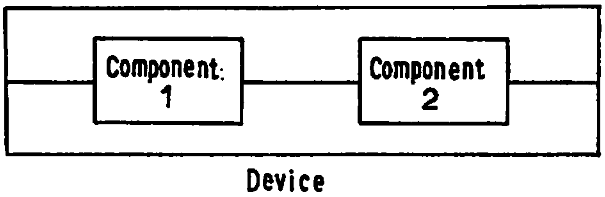

Fig. 1 Two components in series,

The times between failures of components one and two will be independent renewal with increments $\{X_n;\ n\geq 1\}$ and $\{Y_m;\ m\geq 1\}$ respectively. As before we denote ${}^1T_j=\sum_{n=1}^{j}X_n$ and ${}^2T_k=\sum_{m=1}^{k}Y_m$. With these notations $\{{}^1T_j;\ j\geq 0\}$ and $\{{}^2T_k;\ k\geq 0\}$ are respectively the sequences of times of the (j)th and (k)th failures of components one and two. In the superposed process the (r)th event or failure will be denoted by U_r. Thus the process of superposing, simply the cumulative times of failure, is $\{U_r;\ r\geq 0\}$.

Department of Industrial Engineering.

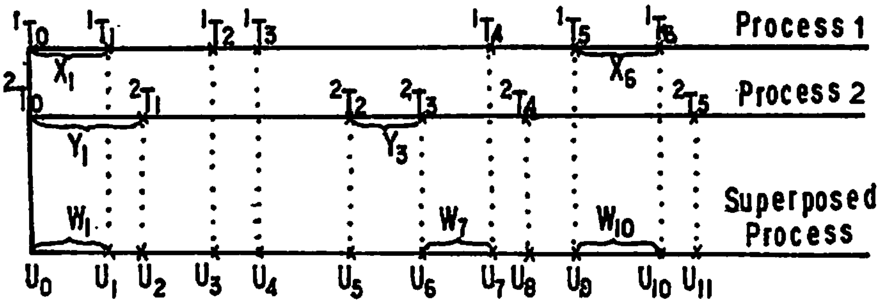

Fig. 2 The superposition of two renewal processes.

The time between failures regardless of type is W_k. Clearly $W_k=U_k-U_{k-1}$. The process is illustrated in Fig. 2.

We shall assume first that $\{X_n;\ n\geq 1\}$ and $\{Y_n;\ m\geq 1\}$ are renewal processes consisting of mutually independent and indentically distributed lifetimes with continuous probability density functions $f(x)$ and $g(y)$ respectively. The cumulative distribution functions will be denoted by $F(x)$ and $G(y)$, the complementary cumulative distribution functions by $\bar{F}(x)$ and $\bar{G}(y)$. We shall further assume that the density functions are positive throughout the positive real line, and that the expectation of the random variables X_n and Y_m are finite.

An event in the superposed process will be described by an ordered pair of random variables $(Z,\ V)$ where Z will equal one or two depending upon the process which produced the event or equivalently, the process to which the event belonged before superposition gives the value of Z. The random variable V will be a non-negative real number: the time elapsed since the last event of the component process which did not produce the event in question in the superposed process. The notation is illustrated in Fig. 3.

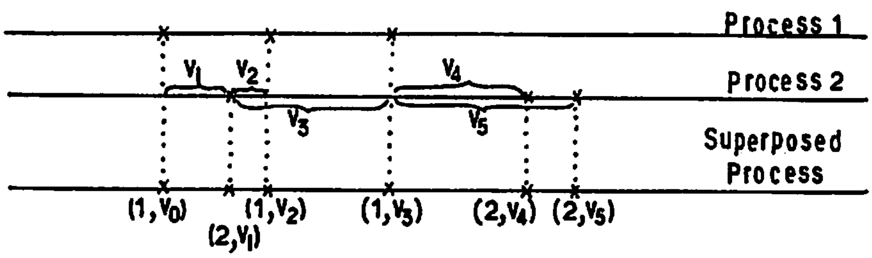

Fig. 3 The superposed process.

Finally we denote by W_n the random variable representing the time between the $(n-1)$ st and (n)th events in superposed process with $W_0=0$. With above notation Cherry, W. P. [2] proved the next theorems.

Theorem 2.1:

The stochastic process $\{Z_n,\ V_n,\ W_n;\ n\geq 0\}$ *is a Markov renewal process defined on*

$$(\{1,\ 2\}\times R^{+},\ (2^{\{1,2\}}\times R^{+}))$$

Next theorem deals with the limiting probability distribution of the semi-Markov process $\{Z(t),\ V(t);\ t\geq 0\}$ associated with the Markov renewal process $\{Z_n,\ V_n,\ U_n;\ n\geq 0\}$.

Theorem 2.2:

$$\lim_{t\to\infty} P_r\{Z(t)=1,\ V(t)\geq v\,|\,Z_0=i,\ V_0=v_0\}$$
$$=\int_{z=0}^{\infty}\int_{u=v}^{\infty}\frac{\bar{F}(z)\cdot\bar{G}(z+u)}{\mu_X\cdot\mu_Y}dudz \tag{2.1}$$
$$\lim_{t\to\infty} P_r\{Z(t)=2,\ V(t)\geq v\,|\,Z_0=i,\ V_0=v_0\}$$
$$=\int_{z=0}^{\infty}\int_{u=v}^{\infty}\frac{\bar{F}(z+u)\bar{G}(z)}{\mu_X\cdot\mu_Y}dudz \tag{2.2}$$

where $E(X_n)=\mu_X,\ E(Y_m)=\mu_Y.$

If we set $v=0$ then the imiting probability (2.1) and (2.2) are the ones with which the process 1 and 2 enter the superposed process respectively.

To illustrate the above limiting results, suppose

$$f(x)=\rho e^{-\rho x},\quad x\geq 0,$$
$$=0,\quad \text{elsewhere;}$$

and

$$g(y)=\nu e^{-\nu y},\quad y\geq 0,$$
$$=0,\quad \text{elsewhere.}$$

We obtain:

$$\lim_{t\to\infty} P_r\{Z(t)=1,\ V(t)\geq 0\,|\,Z_0=i,\ V_0=v_0\}=\frac{\rho}{\nu+\rho},$$

and

$$\lim_{t\to\infty} P_r\{Z(t)=2,\ V(t)\geq 0\,|\,Z_0=i,\ V_0=v_0\}=\frac{\nu}{\nu+\rho}.$$

We set p_1 and p_2 when we set $v=0$ in (2.1) and (2.2) respectively.

3. The mean time up to the (r)th renewal

If U_r denotes the time up to the (r)th event in superposed process, the properties of U_r are not directly obtainable. We next derive the asymptotic mean of U_r according to the method of Cox [1]. Consider the system at the instant U_r. The total time for which all components have then been in use is $2U_r$. But one of the components have still not failed. If we allowed the one to continue untill failure, we would then have obtained the full 'live' of the one, this total time having approximately an expectation,

$$p_1\{p_1 r\mu_X+(p_2 r+1)\mu_Y\}+p_2\{(p_1 r+1)\mu_X+p_2 r\mu_Y\}. \tag{2.3}$$

Thus

$$2E(U_r)=p_1\{p_1 r\mu_X+(p_2 r+1)\mu_Y\}+p_2\{(p_1 r+1)\mu_X+p_2 r\mu_Y\}$$
$$-\{\text{expected forward recurrence-time}\}, \tag{2.4}$$

where the forward recurrence-time is that of one of the components not forming the (r)th failure.

If r is not small, it is reasonable to approximate by the expected limiting recurrence-time

$$\frac{1}{2}\left(\frac{\mu_X^2+\sigma_X^2}{\mu_X}\right)p_2+\frac{1}{2}\left(\frac{\mu_Y^2+\sigma_Y^2}{\mu_Y}\right)p_1. \tag{2.5}$$

Thus

$$E(U_r)=\frac{p_1}{2}\{p_1r\mu_X+(p_2r+1)\mu_Y\}+\frac{p_2}{2}\{(p_1r+1)\mu_X+p_2r\mu_Y\}$$
$$-\frac{1}{2}\left\{\frac{1}{2}\left(\frac{\mu_X^2+\sigma_X^2}{\mu_X}\right)p_2+\frac{1}{2}\left(\frac{\mu_Y^2+\sigma_Y^2}{\mu_Y}\right)p_1\right\}$$
$$=\frac{r}{2}\{p_2\{\mu_Y-\mu_X)+\mu_X\}-\frac{1}{2}\{p_2(\mu_Y-\mu_X)-\mu_Y\}$$
$$-\frac{1}{4}\left\{\mu_Y+\frac{\sigma_Y^2}{\mu_Y}-p_2(\mu_Y-\mu_X)-p_2\left(\frac{\sigma_Y^2}{\mu_Y}-\frac{\sigma_X^2}{\mu_X}\right)\right\}. \tag{2.6}$$

If we set $\mu=\mu_X=\mu_Y$, $\sigma^2=\sigma_X^2=\sigma_Y^2$ (Cox's case), we get,

$$E(U_r)=\frac{r\mu}{2}+\frac{(\mu^2-\sigma^2)}{4\mu}, \tag{2.7}$$

And (2,7) agrees with the Cox's results.

4. Numerical Examples

(i) Exponential distribution.

$$f(x)=\rho e^{-\rho x},\ x\geq 0,$$
$$=0,\ \text{elsewhere};$$

and

$$g(y)=\nu e^{-\nu y},\ y\geq 0,$$
$$0,\ \text{elsewhere}.$$

The means and standard deviations are $\mu_X=\sigma_X=1/\rho$ and $\mu_Y=\sigma_Y=1/\nu$ respectively. And we thus get

$$p_1=\frac{\rho}{\nu+\rho},\quad p_2=\frac{\nu}{\nu+\rho},$$

and in this case we get

$$E(U_r)=\frac{r}{\nu+\rho}.$$

(ii) Special Erlangian distribution with two stages.

$$f(x)=\rho^2xe^{-\rho x},\ x\geq 0.$$
$$=0,\ \text{elsewhere};$$

and

$$g(y)=\nu^2ye^{-\nu y},\ y\geq 0,$$
$$=0,\ \text{elsewhere}.$$

The means and standard deviations are

$$\mu_X=\frac{2}{\rho},\ \sigma_X^2=\frac{2}{\rho^2},\ \mu_Y=\frac{2}{\nu},\ \sigma_Y^2=\frac{2}{\nu^2},$$

respectively. And we thus get

$$p_1=\frac{\rho}{4}\left\{\frac{2}{(\rho+\nu)}+\frac{2\rho+\nu}{(\rho+\nu)^2}+\frac{2\rho\nu}{(\rho+\nu)^3}\right\},$$

and

$$p_2=\frac{\nu}{4}\left\{\frac{2}{(\rho+\nu)}+\frac{\rho+2\nu}{(\rho+\nu)^2}+\frac{2\nu\rho}{(\rho+\nu)^3}\right\}.$$

Then we get in this case

$$E(U_r)\simeq\frac{2}{\rho+\nu}r-\frac{(\rho-\nu)^2}{4(\rho+\nu)^3}r+\frac{1}{4}\left\{\frac{\rho+\nu}{\rho\nu}-\frac{7}{4}\frac{1}{(\rho+\nu)}-\frac{\rho\nu}{(\rho+\nu)^3}\right\}.$$

References

1) Cox, D.R., (1962): Renal Theory, London, Muthuen.
2) Cherry, W.P., (1972): "The Superposition of Two Independent Markov Renewal Processes," *Industria Engineering Technical Reports, The University of Michigan.*