Localization of hereditary rings

Ikeyama, Tamotsu

(出版者 / Publisher)
法政大学工学部
(雑誌名 / Journal or Publication Title)
法政大学工学部研究集報 / 法政大学工学部研究集報
(巻 / Volume)
41
(開始ページ / Start Page)
35
(終了ページ / End Page)
38
(発行年 / Year)
2005-03
(URL)
https://doi.org/10.15002/00003761

Localization of hereditary rings

Tamotsu IKEYAMA*

Localization at a torsion theory was introduced and considered when the torsion theory is hereditary. (See e.g. J. S. Golan [2] or B. Stenström [7].) The definition of the localization is categorical and hence it can be canonically generalized in the notion of the localization at a torsion theory. (See e.g. H. Tachikawa and K. Ohtake [6].) Among other results in T. Ikeyama [4], it is shown that the localization at a hereditary torsion theory and the localization at a torsion theory are different and that these notions, however, coincide when the category is the the one of right R-modules over an arbitrary right hereditary ring. The latter result was proved using the notion of the full subcategory of the local objects and functors. In this paper, we give another proof using ring theoretic notions.

***Key Words** : localization, torsion theory, hereditary ring.*

1. NOTATIONS AND DEFINITIONS.

In this paper we consider the localization at a torsion theory for the category mod-R over a ring R. In the rest of this paper, we suppose that every ring R has the identity element, that every module is unital and that mod-R denotes the category of right R-modules over a ring R. For each right R-module M in the category mod-R, $E(M)$ denotes the injective hull of the module M. Following S. E. Dickson [1], a pair $(\mathcal{T}, \mathcal{F})$ of classes $\mathcal{T}$ and $\mathcal{F}$ of right R-modules is said to be a torsion theory for the category mod-R provided :

(T1) $\mathcal{T} \cap \mathcal{F} = \{0\}$.

(T2) If a module T is in the class $\mathcal{T}$ and $T \longrightarrow M \longrightarrow 0$ is an exact sequense in the category mod-R of right R-modules, then M is in the class $\mathcal{T}$.

(T3) If a module F is in the class $\mathcal{F}$ and $0 \longrightarrow M \longrightarrow F$ is an exact sequense in the category mod-R of right R-modules, then M is in the class $\mathcal{F}$.

(T4) For each right R-module M, there exists an exact sequence

$$0 \longrightarrow T \longrightarrow M \longrightarrow F \longrightarrow 0$$

such that the module T is in the class $\mathcal{T}$ and the module F is in the class $\mathcal{F}$.

Moreover, the image of the homomorphism $T \longrightarrow M$ in the condition (T4) is called the torsion submodule of the right R-module M. For basic properties of torsion theories see S. E. Dickson [1], J. S. Golan [2] or B. Stenström [7]. A torsion theory $(\mathcal{T}, \mathcal{F})$ for mod-R is said to be hereditary provided the class $\mathcal{T}$ is closed under taking submodules. We note that a torsion theory in J. S. Golan [2] means a hereditary torsion theory.

Let $(\mathcal{T}, \mathcal{F})$ be a torsion theory for mod-R in the sence of S. E. Dickson [1]. Then, a right R-module L is said to be $\mathcal{T}$-injective provided the functor $\mathrm{Hom}_R(-, L)$ is exact on all exact sequences

$$0 \longrightarrow X \longrightarrow Y \longrightarrow T \longrightarrow 0$$

with the right R-module T in the class $\mathcal{T}$. The localization of a right R-module M at the torsion theory $(\mathcal{T}, \mathcal{F})$ means the homomorphism

$$f : M \longrightarrow L$$

with the properties :

(L1) $\mathrm{Ker}(f)$ and $\mathrm{Coker}(f)$ are in the class $\mathcal{T}$.

(L2) L is in the class $\mathcal{F}$.

(L3) L is $\mathcal{T}$-injective.

To consider the localization at a torsion theory for the category mod-R of right R-modules and the localization at a hereditary torsion theory for mod-R, we use the following notations. Throughout this paper, $(\mathcal{T}, \mathcal{F})$ means a torsion theory for mod-R over a ring R and $\mathcal{E}$ denotes the sub class of $\mathcal{F}$ such that

$$\{E \in \mathcal{F} \mid E \text{ is injective}\}.$$

Using these notations, we define the torsion theory $(\mathcal{T}', \mathcal{F}')$ for the the category mod-R of right R-modules having the property that a right R-module M is in the class $\mathcal{T}'$ if and only if the module M satisfies the condition

$$\mathrm{Hom}_R(M, E) = 0$$

for every right R-module E in the class $\mathcal{E}$. Then it is obvious that the torsion theory $(\mathcal{T}', \mathcal{F}')$ for mod-R is herediraty and that the class $\mathcal{T}$ is contained in the class $\mathcal{T}'$. The

* College of Engineering

hereditary torsion theory $(\mathcal{T}', \mathcal{F}')$ for mod-R is said to be cogenerated by the class $\mathcal{E}$.

We also denote

$$\mathcal{L} = \{L \in \mathcal{F} \mid L \text{ is } \mathcal{T}\text{-injective}\}$$

and

$$\mathcal{L}' = \{L \in \mathcal{F}' \mid L \text{ is } \mathcal{T}'\text{-injective}\}.$$

Using these notations, we consider the localization at a torsion theory $(\mathcal{T}, \mathcal{F})$ for the the category mod-R of right R-modules and the localization at the hereditary torsion theory $(\mathcal{T}', \mathcal{F}')$ for mod-R.

2. PRELIMINARY.

According to the definition of the hereditary torsion theory $(\mathcal{T}', \mathcal{F}')$ for the the category mod-R of right R-modules cogenerated by the class $\mathcal{E}$, the class $\mathcal{T}$ is contained in the class $\mathcal{T}'$. Hence, T. Ikeyama [4, Lemma 17] implies that if the class $\mathcal{L}$ is contained in the class $\mathcal{L}'$, then every localization of a right R-module at the torsion theory $(\mathcal{T}, \mathcal{F})$ for the the category mod-R of right R-modules turns out to be the localization at the hereditary torsion theory $(\mathcal{T}', \mathcal{F}')$ for mod-R.

Thus, if the class $\mathcal{L}$ is always contained in the class $\mathcal{L}'$, then each localization at any torsion theory for mod-R coincides with the localization at some hereditary torsion theory for mod-R. The statement, however, is not true in general. We first give such an example.

Example 1. Let K be a field,

$$S = \left\{ \begin{pmatrix} a & K & K & K \\ 0 & K & K & K \\ 0 & 0 & a & K \\ 0 & 0 & 0 & K \end{pmatrix} \middle| \ a \in K \right\}$$

the subring of the ring of 4×4 matrices over the field K. Consider the right S-modules

$$M = \begin{pmatrix} 0 & 0 & 0 & K \end{pmatrix},$$

$$L = \begin{pmatrix} 0 & K & K & K \end{pmatrix}$$

and

$$E = \begin{pmatrix} K & K & K & K \end{pmatrix}$$

by matrices operations. Then, the module $E \oplus (E/L)$ cogenerates the torsion theory $(\mathcal{T}, \mathcal{F})$ for the the category mod-S of right S-modules. Concerning the torsion theory $(\mathcal{T}, \mathcal{F})$ for mod-S, it is proved in T. Ikeyama [3, Example 5] that the canonical inclusion

$$i : M \longrightarrow L$$

has the following properties:

(1) The homomorphism i is the localization at the torsion theory $(\mathcal{T}, \mathcal{F})$ for mod-S.

(2) The homomorphism i is not the localization at any hereditary torsion theory for mod-S.

On the other hand, if the class $\mathcal{L}$ is contained in the class $\mathcal{L}'$, then every localization of a module at the torsion theory $(\mathcal{T}, \mathcal{F})$ for mod-S turns out to be the localization at the hereditary torsion theory $(\mathcal{T}', \mathcal{F}')$ for mod-S by T. Ikeyama [4, Lemma 17]. Therefore the class $\mathcal{L}$ does not contained in the class $\mathcal{L}'$.

In the rest of this paper we show that, if the ring R is right hereditary, then the localization at the torsion theory $(\mathcal{T}, \mathcal{F})$ for the category mod-R of right R-modules turns out to be the localization at the hereditary torsion theory $(\mathcal{T}', \mathcal{F}')$ for mod-R.

3. LOCALIZATION OVER ARBITRARY RINGS.

We consider the right R-modules in the class $\mathcal{L}$ over an arbitrary ring R and obtain the following result.

Lemma 2. If the right R-module L is in the class $\mathcal{L}$, then the injective hull $E(L)$ of the module L is also in the class $\mathcal{L}$.

Proof. It is obvious that the identity homomorphism

$$\mathrm{id}_L : L \longrightarrow L$$

satisfies the definition of the localization of the module L at the torsion theory $(\mathcal{T}, \mathcal{F})$ for the category mod-R of right R-modules. Let T be the torsion submodule of the module L. Then, the fact that the module L has the localization at the torsion theory $(\mathcal{T}, \mathcal{F})$ for mod-R induces that the injective hull $E(L/T)$ of the right R-module L/T is in the class $\mathcal{F}$ by T. Kato and T. Ikeyama [5, Lemma 2.9]. On the other hand, the module L is in the class $\mathcal{L}$ which is a sub class of the class $\mathcal{F}$. Since the class $\mathcal{F}$ is closed under taking submodules, the submodule T of L is also in the class $\mathcal{F}$. This result induces that the module T is in the class $\mathcal{T}$ and is in the class $\mathcal{F}$. Hence we have that $T = 0$ for $\mathcal{T} \cap \mathcal{F} = \{0\}$. Thus, we obtain that the injective module $E(L)$ is in the class $\mathcal{F}$, for the module is equal to $E(L/T)$ which is in the class $\mathcal{F}$. Therefore, the module $E(L)$ is in the class $\mathcal{L}$, since every injective module is $\mathcal{T}$-injective.

For each right R-module L in the class $\mathcal{L}$, we also have the following lemma.

Lemma 3. If the right R-module L is in the class $\mathcal{L}$, it is in the class $\mathcal{F}'$.

Proof. According to the preceding lemma, the injective hull $E(L)$ of the module L is in the class $\mathcal{F}$. Hence, $E(L)$

is in the class $\mathcal{E}$ for every injective module is $\mathcal{T}$-injective. According to the definition of the hereditary torsion theory $(\mathcal{T}', \mathcal{F}')$ for mod-R cogenerated by the class $\mathcal{E}$, the class $\mathcal{E}$ is contained in the class $\mathcal{F}'$. Thus the module $E(L)$ is in the class $\mathcal{F}'$. Therefore, the module L is also in the class $\mathcal{F}'$ since the class $\mathcal{F}'$ is closed under taking submodules.

For a right R-module L in the class $\mathcal{L}$ with the injective hull $E(L)$, the factor module $E(L)/L$ satisfies the following condition.

Lemma 4. If the right R-module L is in the class $\mathcal{L}$, then the factor module $E(L)/L$ is in the class $\mathcal{F}$.

Proof. Let X be the submodule of $E(L)$ such that X/L is the torsion submodule of the module $E(L)/L$ for the torsion theory $(\mathcal{T}, \mathcal{F})$ for mod-R. Then, the canonical exact sequence

$$0 \longrightarrow L \longrightarrow X \longrightarrow X/L \longrightarrow 0$$

splits for L is $\mathcal{T}$-injective. Thus, $X = L \oplus Y$ with some submodule Y of $E(L)$. This means that $Y = 0$, since the submodule L is essential in $E(L)$. Therefore, we have that $X/L = 0$ for it is isomorphic to Y. Hence the module $E(L)/L$ is in the class $\mathcal{F}$.

4. LOCALIZATION OVER RIGHT HEREDITARY RINGS.

In the rest of this paper, we consider the localization over right hereditary rings. Note that torsion theories for the the category mod-R of right R-modules over a right hereditary ring R need not be hereditary in general. Following example was given in S. E. Dickson [1].

Example 5. Let $\mathbb{Z}$ be the ring of integers. Then the ring $\mathbb{Z}$ is right hereditary. We describe the class of injective right $\mathbb{Z}$-modules with the notation $\mathcal{T}$. Then the class $\mathcal{T}$ induces the class $\mathcal{F}$ of right $\mathbb{Z}$-modules such that a right $\mathbb{Z}$-module M is in the class $\mathcal{F}$ if and only if the module M satisfies the condition

$$\mathrm{Hom}_{\mathbb{Z}}(T, M) = 0$$

for every right $\mathbb{Z}$-module T which is contained in the class $\mathcal{T}$. According to S. E. Dickson [1], the pair of classes $(\mathcal{T}, \mathcal{F})$ is a torsion theory for mod-$\mathbb{Z}$. It is obvious that the right $\mathbb{Z}$-module $\mathbb{Q}$ of rational numbers is in the class $\mathcal{T}$ and that $\mathbb{Z}$ is not in the class $\mathcal{T}$. Hence the class $\mathcal{T}$ is not closed under taking submodules. Therefore, the torsion theory $(\mathcal{T}, \mathcal{F})$ for mod-$\mathbb{Z}$ is not hereditary.

We consider right R-modules in the class $\mathcal{L}$ for a torsion theory $(\mathcal{T}, \mathcal{F})$ for the the category mod-R of right R-modules over a right hereditary ring R and obtain the following result.

Lemma 6. Let R be a right hereditary ring and $(\mathcal{T}, \mathcal{F})$ a torsion theory for the category mod-R of right R-modules. If the right R-module L is in the class $\mathcal{L}$, then the right R-module $E(L)/L$ is in the class $\mathcal{F}'$ where $(\mathcal{T}', \mathcal{F}')$ is the hereditary torsion theory for mod-R cogenerated by the class $\mathcal{E}$.

Proof. Since the ring R is right hereditary, the module $E(L)/L$ is injective for every L in the class $\mathcal{L}$. Moreover, the module $E(L)/L$ is in the class $\mathcal{F}$ by the preceding lemma. Hence, the module $E(L)/L$ is in the class $\mathcal{E}$ which is contained in the class $\mathcal{F}'$.

In the following proposition, for a torsion theory $(\mathcal{T}, \mathcal{F})$ for mod-R over a right hereditary ring R we show that each right R-module L in the class $\mathcal{L}$ is $\mathcal{T}'$-injective where $(\mathcal{T}', \mathcal{F}')$ is the hereditary torsion theory cogenerated by the class $\mathcal{E}$.

Proposition 7. Let R be a right hereditary ring and $(\mathcal{T}, \mathcal{F})$ a torsion theory for the the category mod-R of right R-modules. If the right R-module L is in the class $\mathcal{L}$, then L is $\mathcal{T}'$-injective where $(\mathcal{T}', \mathcal{F}')$ is the hereditary torsion theory for mod-R cogenerated by the class $\mathcal{E}$.

Proof. Let

$$f : X \longrightarrow L$$

be a homomorphism and

$$0 \longrightarrow X \xrightarrow{\ g\ } Y \xrightarrow{\ h\ } T' \longrightarrow 0$$

an exact sequence with a right R-module T' which is in the class $\mathcal{T}'$. Then the morphism f and the exact sequence induce the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{\ g\ } & Y & \xrightarrow{\ h\ } & T' & \longrightarrow & 0 \\ & & {\scriptstyle f}\downarrow & & & & & & \\ 0 & \longrightarrow & L & \xrightarrow{\ i\ } & E(L) & \xrightarrow{\ j\ } & E(L)/L & \longrightarrow & 0 \end{array}$$

with the inclusion i and the canonical surjection j. Since the module $E(L)$ is injective, there exists a homomorphism

$$f_1 : Y \longrightarrow E(L)$$

such that $f_1\, g = i\, f$. The homomorphism f_1 induces the homomorphism

$$f_2 : T' \longrightarrow E(L)/L$$

with the condition $f_2\, h = j\, f_1$. Now, we have the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{g} & Y & \xrightarrow{h} & T' & \longrightarrow & 0 \\ & & f\downarrow & & f_1\downarrow & & f_2\downarrow & & \\ 0 & \longrightarrow & L & \xrightarrow{i} & E(L) & \xrightarrow{j} & E(L)/L & \longrightarrow & 0 \end{array}$$

with exact rows. According to the preceding lemma, the module $E(L)/L$ is in the class $\mathcal{F}'$. Hence, we have that $f_2 = 0$ since the module T' is in the class $\mathcal{T}'$. It means that $j\, f_1$ is also 0. This result induces the existence of the homomorphism

$$f_3 : Y \longrightarrow L$$

which makes the lower triangle of the following diagram

$$\begin{array}{ccccc} & & X & \xrightarrow{g} & Y \\ & & f\downarrow & f_3 \swarrow & f_1\downarrow \\ 0 & \longrightarrow & L & \xrightarrow{i} & E(L) \end{array}$$

commutative. Then $i\, f_3\, g = f_1\, g = i\, f$ implies $f_3\, g = f$, for the homomorphism i is monomorphic. Therefore, the module L is $\mathcal{T}'$-injective as desired.

Using the above results, we give a ring-theoretic proof for T. Ikeyama [4, Corollary 22] without using the notion of functors.

Theorem 8. Let R be a right hereditary ring and $(\mathcal{T}, \mathcal{F})$ a torsion theory for the category mod-R of right R-modules. Then the localization at the torsion theory $(\mathcal{T}, \mathcal{F})$ for mod-R turns out to be the localization at the hereditary torsion theory $(\mathcal{T}', \mathcal{F}')$ for mod-R where $(\mathcal{T}', \mathcal{F}')$ is the hereditary torsion theory cogenerated by the class $\mathcal{E}$.

Proof. Since the definition of the hereditary torsion theory $(\mathcal{T}', \mathcal{F}')$ for mod-R, the class $\mathcal{T}$ is contained in the class $\mathcal{T}'$. Moreover, the preceding proposition implies that each right R-module L in the class $\mathcal{L}$ is $\mathcal{T}'$-injective. On the other hand, the module L is in $\mathcal{F}'$ by Lemma 3. Thus, we obtain that the class $\mathcal{L}$ is contained in the class $\mathcal{L}'$. Therefore, we have conditions that he class $\mathcal{T}$ is contained in the class $\mathcal{T}'$ and that the class $\mathcal{L}$ is contained in the class $\mathcal{L}'$. Thus, every localization at the torsion theory $(\mathcal{T}, \mathcal{F})$ for mod-R turns out to be the localization at the hereditary torsion theory $(\mathcal{T}', \mathcal{F}')$ for mod-R by T. Ikeyama [4, Lemma 17].

REFERENCES

1) S. E. Dickson, A torsion theory for abelian categories, Trans. Amer. Math. Soc., 121 (1966), 223-235.

2) J. S. Golan, Torsion Theories, Pitman Monographs and Surveys, Vol. 29, Longman, 1986.

3) T. Ikeyama, Localizations with respect to non-hereditary torsion theories, Tsukuba J. Math., 7 (1983), 185-189.

4) T. Ikeyama, On the localization of a ring at torsion theories, Comm. Algebra, 28 (2000), 2801-2815.

5) T. kato and T. Ikeyama, Localization in abelian categories, Comm. Algebra, 18 (1990), 2519-2540.

6) H. Tachikawa and K. Ohtake, Colocalization and localization in abelian categories, J. Algebra, 56 (1979), 1-23.

7) B. Stenström, Rings of Quotients, Springer Verlag, 1975.