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(出版者 / Publisher)
法政大学文学部
(雑誌名 / Journal or Publication Title)
Bulletin of Faculty of Letters, Hosei University / 法政大学文学部紀要
(巻 / Volume)
50
(開始ページ / Start Page)
1
(終了ページ / End Page)
5
(発行年 / Year)
2005-03-01
(URL)
https://doi.org/10.15002/00002925

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1 Introduction

There are some different ways for formalizing classical logic. One of those uses Peirce's rule, which states that if $A \supset B$ implies A, then we have A without the assumption $A \supset B$. This rule contains only implication as the logical connectives, therefore it is suitable for the implicational fragment of classical logic. Komori [3] introduced a system of calculus named $\lambda\rho$-calculus, whose type assignment system corresponds with the implicational fragment of classical natural deduction formalized with Peirce's rule. He also defined a set of contractions on $\lambda\rho$-calculus, and stated the normalization theorem in his paper mentioned above. But according to his recent research [4], any direct proof of the weak normalization theorem has not yet known, although he already proved the strong one. In this note, we prove the weak normalization theorem for Komori's $\lambda\rho$-calculus with the definition of an appropriate degree of deductions. The proof is similar to that of our former paper [1] for full classical natural deduction formalized with classical absurdity rule. To define the degree of deductions for $\lambda\rho$-calculus, which measures redexes, we have to find segment of formula-occurrences as in the case of intuitionistic logic by Prawitz [5, 6]. We use Hindley's notations [2] for λ-calculus basically, and those of Komori for $\lambda\rho$-calculus.

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2 Komori's $\lambda\rho$-calculus

In this section, we state some basic definitions of Komori's $\lambda\rho$-calculus according his paper [3]. See it for more details.

Definition ($\lambda\rho$-terms) Two infinite sequences of variables are assumed to be given, named λ-variables and ρ-variables, and $\lambda\rho$-terms are defined inductively as follows.

- Each λ-variable is a $\lambda\rho$-term.
- If M and N are $\lambda\rho$-terms, then (MN) is a $\lambda\rho$-term.
- If a is a ρ-variable and M is a $\lambda\rho$-term, then (aM) is a $\lambda\rho$-term.
- If x is a λ-variable and M is a $\lambda\rho$-term, then $(\lambda x.M)$ is a $\lambda\rho$-term.
- If a is a ρ-variable and M is a $\lambda\rho$-term, then $(\rho a.M)$ is a $\lambda\rho$-term.

Definition ($\rho\beta$-contractions) $\rho\beta$-contractions, or re-write rules from $\rho\beta$-redexes to their contractums are defined as follows.

- $aMN \rhd_{1\rho\beta} aM$
- $(\lambda x.M)N \rhd_{1\rho\beta} [N/x]M$
- $(\rho a.M)N \rhd_{1\rho\beta} \lambda a.([\lambda x.a(xN)/a]M)N$

Definition (Types) An infinite sequence of type-variables is assumed to be given, and types are defined inductively as follows.

- Each type-variable is a type.
- If σ and τ are types, then $(\sigma \to \tau)$ is a type.

Definition (The system $TA_{\lambda\rho}$) The type-assignment system $TA_{\lambda\rho}$ is defined as follows.

Axiom of $TA_{\lambda\rho}$:

$$x : \tau \mapsto x : \tau$$

Deduction-rules of $TA_{\lambda\rho}$:

$$\frac{\langle \Gamma_1; P : (\sigma \to \tau)\rangle\langle \Gamma_2; Q : \sigma\rangle}{\langle \Gamma_1 \cup \Gamma_2; (PQ) : \tau\rangle}\ (\to E) \quad [\text{if } \Gamma_1 \cup \Gamma_2 \text{ is consistent}],$$

$$\frac{\langle \Gamma; P : \tau\rangle}{\langle \Gamma - x; (\lambda x.P) : \sigma \to \tau\rangle}\ (\to I) \quad [\text{if } \Gamma \text{ is consistent with } x : \sigma],$$

$$\frac{\langle \Gamma; P : \tau\rangle}{\langle \Gamma, a : \tau; (a.P) : \sigma\rangle}\ (Absurd) \quad [\text{if } \Gamma \text{ is consistent with } a : \tau],$$

$$\frac{\langle \Gamma; P : \tau\rangle}{\langle \Gamma - a; (\rho a.P) : \tau\rangle}\ (Rati) \quad [\text{if } \Gamma \text{ is consistent with } a : \tau].$$

3 normalization

In order to define the degree of $TA_{\lambda\rho}$-deduction used in the proof of the normalization theorem, we first introduce a measure for terms, which represents the maximal length of the corresponding segment in natural deduction.

Definition (rank of terms) For each $\lambda\rho$-term P, we define the *rank* $r(P)$ inductively as follows.

- If P is of the form x, MN, aM, or $\lambda x.M$, then $r(P)$ is 1.
- If P is of the form $\rho a.M$, then $r(P)$ is

$$\max\Big\{r(M), \max_{Q}\{r(Q)\}\Big\} + 1,$$

where Q varies all terms such that aQ is a subterm of M.

Definition (degree of $TA_{\lambda\rho}$-deductions) Let Δ be a $TA_{\lambda\rho}$-deduction of the $TA_{\lambda\rho}$-formula $\left\langle \Gamma, T, \tau \right\rangle$. First, we define an ordered pair of natural numbers denoted by $d_0(\Delta)$ as follows. If M is $\rho\beta$-normal then $d_0(\Delta)$ is $\langle 0, 0\rangle$, otherwise,

$$d_0(\Delta) = \max\left\{ \langle |\delta|, r(P)\rangle \;\middle|\; \underline{PN} \text{ is a } \rho\beta\text{-redex-occurrence in } T \text{ and } \underline{P} \text{ has type } \delta \text{ in } \Delta. \right\},$$

where $|\delta|$ is the length of δ, defined in [2], that is the total number of occurrences of type-variables in δ. Two ordered pairs as the values of d_0 are compared by lexicographical order. Next we define a natural number denoted $d_1(\Delta)$ as the number of $\rho\beta$-redex-occurrences, say $\underline{PN}$, in T, which satisfies $\langle |\delta|, r(P)\rangle = d_0(\Delta)$ where $\underline{P}$ has type δ in Δ. Finally, we define the degree of Δ denoted by $D(\Delta)$ as below.

$$D(\Delta) = \langle d_0(\Delta), d_1(\Delta)\rangle$$

We also compare two values of D by lexicographical order.

Theorem (Normalizabilty of typable terms) *If a $\lambda\rho$-term has a type in $TA_{\lambda\rho}$, then it is $\rho\beta$-normalizable.*

Proof Let Δ be a given $TA_{\lambda\rho}$-deduction of $\left\langle \Gamma, T, \tau \right\rangle$ which is not $\rho\beta$-normal. We call E the set of all $\rho\beta$-redex-occurrences $\underline{PN}$ in M which satisfies that $\langle |\delta|, r(P)\rangle$ is equal to $d_0(\Delta)$ where $\underline{P}$ has type δ in Δ. Take one element of E, say $\underline{P_0N_0}$, such that there is no element of E in $\underline{N_0}$. Suppose T $\rho\beta$-contracts to T_1 by replacing $\underline{P_0N_0}$ by its contractum, that is, $\langle T, \underline{P_0N_0}, T_1\rangle$. If P is of the form aM or $\lambda x.M$, then define $T' \equiv T_1$. Otherwise, that is P_0 is a ρ-abstract, say $\rho a.M$, execute a $\rho\beta$-contraction of T_1 with one of the redexes created by the substitution for a in M if there is an occurrence of a in M, and by repeating such $\rho\beta$-contraction, take one $\rho\beta$-reduction

$$\langle T_1, \underline{R_1}, T_2\rangle, \langle T_2, \underline{R_2}, T_3\rangle, \ldots, \langle T_n, \underline{R_n}, T_{n+1}\rangle$$

if there are exactly n occurrences of a in M. In this case, define $T' \equiv T_{n+1}$. Let Δ' be the $TA_{\lambda\rho}$-deduction of $\left\langle \Gamma', T', \tau \right\rangle$ corresponding with Δ. Then we have $d(\Delta) > d(\Delta')$ by definition. Therefore, by transfinite induction on ω^3, it holds that every typable $\lambda\rho$-term is $\rho\beta$-normalizable.

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