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安東, 祐希 / ANDOU, Yuuki

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ANDOU Yuuki*

1 Introduction

Gentzen defined in [4] the logical system called *natural deduction* and he introduced his Hauptsatz. But it was proved in the system of sequent-calculus, because his system of natural deduction was not suitable for the poof of the Hauptsatz in the case of classical logic. After the Gentzen's work, Prawitz [5] reorganized Gentzen's original natural deduction, and he proved normalization theorem of some systems of natural deduction, including a restricted classical logic, that is another representation of Gentzen's Hauptsatz. For the full system of classical natural deduction, we proved the normalization theorem in our previous work [1]. Here we call a system full if it has all logical connectives primitively. On the other hand, in [6], Prawitz introduced the notion *validity* for the deductions in the systems of natural deduction. He defined this notion by investigating Gentzen's idea that the introduction rules give the meanings of logical connectives. However, Prawitz's system has a restriction of logical connectives in the case of classical logic, so we have to extend his definition of validity to apply it for the full system. In this note, we introduce an extended definition of Prawitz's validity for the full classical natural deduction in the form of *strong validity*. For this extension, we use the system of typed-terms corresponding with the full classical natural deduction introduced in [2, 3]. But the application of the extended definition for the

* Department of Philosophy, Hosei University, Tokyo 102-8160, Japan.
E-mail: `norakuro@i.hosei.ac.jp`

proof of strong normalization theorem is our further work.

2 Classical Natural Deduction

Our system of classical natural deduction is formulated as a system of an extension of typed λ-calculus in Church-style, named λ^C-calculus, which is introduced in our previvous papers [2, 3]. Here we sketch the outline of our system. For more details, see the papers mentioned above.

2.1 Typed-terms

Types are formulas of a first order language $\mathcal{L}$ which contains full logical connectives. We use the following letters, possibly with sub- or superscripts, as metavariables.

- a, b, c : for individual variables of $\mathcal{L}$.
- p, q : for $\mathcal{L}$-terms.
- A, B, C, D : for $\mathcal{L}$-formulas.
- x, y, z : for λ^C-variables.
- r, s, t, u, v, w : for λ^C-terms.

Typed λ^C-terms are defined as follows. They have canonical correspondences with derivations in Prawitz's system C of classical natural deduction without the restriction of primitive logical connectives and of the complexity of the formulae introduced by classical absurdity rules.

- If x is a λ^C-variable and A a $\mathcal{L}$-formula, then (x^A) is a λ^C-term of type A.
- By the following schemata, we explain other constructions of terms.

$$\frac{\begin{matrix}[x^A]\\ t : B\end{matrix}}{(\lambda x^A.t) : A \supset B}\,(\supset I), \qquad \frac{t : A \supset B \quad u : A}{t(u) : B}\,(\supset E),$$

$$\frac{t:A \quad u:B}{\langle t,u\rangle : A\wedge B}\ (\wedge I),\quad \frac{t:A\wedge B}{t(\pi 0):A}\ (\wedge E_0),\quad \frac{t:A\wedge B}{t(\pi 1):B}\ (\wedge E_1),$$

$$\frac{t:A}{\{t,B\}:A\vee B}\ (\vee I_0),\quad \frac{t:B}{\{A,t\}:A\vee B}\ (\vee I_1),$$

$$\frac{t:A\vee B \quad \begin{matrix}[x^A]\\ u:C\end{matrix} \quad \begin{matrix}[y^B]\\ v:C\end{matrix}}{t(\lambda^{\vee}x^A.u,\lambda^{\vee}y^B.v):C}\ (\vee E),$$

$$\frac{t:A}{(\lambda^{\forall}a.t):\forall aA}\ (\forall I),\quad \frac{t:\forall aA}{t(p):A[p/a]}\ (\forall E),$$

$$\frac{t:A[p/a]}{(\sigma_A^{p,a}t):\exists aA}\ (\exists I),\quad \frac{t:\exists aA \quad \begin{matrix}[x^A]\\ u:C\end{matrix}}{t(\lambda^{\exists}x^A.u):C}\ (\exists E),$$

$$\frac{\begin{matrix}[x^{\neg C}]\\ t:\bot\end{matrix}}{(\lambda^{\bot}x^{\neg C}.t):C}\ (\bot_c).$$

Note that we suppose every λ^C-term satisfies $\lambda^{\bot}$-regularity [2, 3].

2.2 Reductions

There are two kinds of contractions for λ^C-terms called *essential* ones and *structural* ones. They are denoted by $\rhd_e$ and $\rhd_s$ respectively.

- $(\lambda x^A.t)(u) \rhd_e t[u/x^A]$
- $\langle t,u\rangle(\pi 0) \rhd_e t, \quad \langle t,u\rangle(\pi 1) \rhd_e u$
- $\{t,B\}(\lambda^{\vee}x^A.u,\lambda^{\vee}y^B.v) \ \rhd_e\ u[t/x^A], \quad \{A,t\}(\lambda^{\vee}x^A.u,\lambda^{\vee}y^B.v) \ \rhd_e\ v[t/y^B]$

- $(\lambda^{\forall} a.t)(p) \rhd_e t[p/a]$
- $(\sigma_A^{p,a} t)(\lambda^{\exists} x^A.u) \rhd_e u[p/a; t/x^A]$
- $t(\lambda^{\vee} x^A.u, \lambda^{\vee} y^B.v)\varepsilon \rhd_s t(\lambda^{\vee} x^A.u\varepsilon, \lambda^{\vee} y^B.v\varepsilon)$
- $t(\lambda^{\exists} x^A.u)\varepsilon \rhd_s t(\lambda^{\exists} x^A.u\varepsilon)$
- $(\lambda^{\perp} x^{\neg C}.t)\varepsilon \rhd_s (\lambda^{\perp} x^{\neg D}.t[\ \varepsilon)_x/\ )_x, x^{\neg D}/x^{\neg C}])$

In the schemata above, ε represents an eliminator [2, 3] from C to D.

We denote $t \succ s$ iff s is obtained from t by replacing an occurrence of a $\rhd_e$-or $\rhd_s$-redex by its contractum, also the relation $\gg$ is defined as the reflexive and transitive closure of $\succ$.

3 Strong Validity

In this section, we define the notion of the strong validity for our system of typed-terms which is an extension of Prawitz's one for natural deduction system of minimal logic, intuitionistic logic, and of classical logic with some restrictions. In our definition, some terminologies introduced in [3] are used, such as segment-tree and accepter. They simplify the denotation of the definition.

Definition (Strong validity) Let s be a λ^C-term. We define that s is *strongly valid* inductively if and only if one of the following conditions holds.

- s is of the form $\langle t, u\rangle$, $\{t, B\}$, $\{A, t\}$ or $(\sigma_A^{p,a} t)$ where t and u are strongly valid.
- s is of the form $(\lambda x^A.t)$ where $t[u/x^A]$ is strongly valid for every strongly valid term u of type A.
- s is of the form $(\lambda^{\forall} a.t)$ where $t[p/a]$ is strongly valid for every $\mathcal{L}$-term p.
- s is of the form $t(u)$, $t(\pi 0)$, $t(\pi 1)$, $t(p)$ or $(\lambda^{\perp} x^{\neg C}.t)$, and s' is strongly valid for all s' satisfying $s \succ s'$.
- s is of the form $t(\lambda^{\vee} x^A.u, \lambda^{\vee} y^B.v)$ where u and v are strongly valid, and s' is strongly valid for all s' satisfying $s \succ s'$. Moreover, if $t \gg t'$ and if $\{w, B\}$ or $\{A, w\}$ is an accepter of a maximal segment-tree from t' in t',

then $u[w/x^A]$ or $v[w/y^B]$ respectively is strongly valid.

- s is of the form $t(\lambda^{\exists}x^A.u)$ where u is strongly valid, and s' is strongly valid for all s' satisfying $s \succ s'$. Moreover, if $t \gg t'$ and if $(\sigma_A^{p,a}w)$ is an accepter of a maximal segment-tree from t' in t', then $u[p/a; w/x^A]$ is strongly valid.

We have the following theorem.

Theorem *If a λ^C-term t is strongly valid, then t is strongly normalizable.*

The theorem can be proved similarly to the corresponding theorem of Prawitz [6] and we omit the details of the proof.

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