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ON THE PASINETTI GROWTH MODEL AND THE ANTI-PASINETTI THEORY (II)

By KOICHI MIYAZAKI

CONTENTS

3. The Shift from the Pasinetti Case to the Anti-Pasinetti Case

In the anti-Pasinetti theory originally proposed by Meade (1966) and Samuelson-Modigliani (1966) (see the *References* of the Part I of this article in the *Keizai Shirin*, Vol. 64, No. 3), it is argued as follows: there can hold the Pasinetti steady states in which the two economic classes, the workers and the capitalists, steadily coexist, if and only if the workers' propensity to save, s_w, is not very high. However, it argues, if s_w becomes higher than a certain critical level, to be denoted by s_w^1, the economy will be bound to converge to a uni-class steady state where the capitalists have disappeared from the scene and there remain only the workers. That is, the anti-Pasinetti steady state will be necessarily attained.

In short, by a sufficient increase in the thrift of the workers, the economy shifts from the Pasinetti steady states to the anti-Pasinetti steady states.

In this section, following such a scenario of the anti-Pasinetti theory, particularly of that written by Samuelson-Modigliani, we will consider the comparative statics of this sort in the more realistic framework of the Balestra-Baranzini model.

Now, concerning the critical value of s_w, we make the following definition.

DEFINITION 8: s_w^1 is defined by $i = f'(k^{**}(s_w^1))$.

Since, by the definition of k^{**}, $k^{**}(s_w)$ is an increasing function of s_w, it will follow, from Theorem 1 in the previous section and by this definition of s_w^1, that $s_w < s_w^1$ is necessary and sufficient for the existence, uniqueness, and global stability of the Pasinetti steady state.

In the following comparative statics, we will compare different steady states with different s_w, with all the other data, such as n, i, s_c, etc., regarded to be constant.

However, for a technical convenience, we make the following assumption concerning the relationship among those data other than s_w.

ASSUMPTION 2: $i \geqq \{k_2 f'(k_2)/f(k_2)\}(n/s_c)$.

This assumption means that the rate of interest, i, is not so small as the proportion of it to the capitalists' equilibrium rate of profit, n/s_c, becomes smaller than $k_2 f'(k_2)/f(k_2)$, that is, the relative share of profit at $k = k_2$.

From the realistic point of view, this assumption may be said to be hardly restrictive, since the ratio of the rate of interest to the capitalists' rate of profit will be ordinarily greater than one half, and scarcely less than one third, whereas the relative profit share is ordinarily much less than one half, and scarcely greater than one third.

PROPOSITION 9: The above assumption is equivalent to $s_w^1 \leqq s_c$.

Proof. By the definition of k_2 and the above definition of s_w^1, we have $k_2 = k^{**}(s_w^1)$. This means $k_2/f(k_2) = s_w^1/n$.

Therefore, the above assumption is equivalent to $i \geqq f'(k_2)(s_w^1/n)(n/s_c) = f'(k_2)(s_w^1/s_c)$. Hence, by $f'(k_2) = i$, it is equivalent to $1 \geqq s_w^1/s_c$, that is, $s_c \geqq s_w^1$. Q. E. D.

The condition, $s_w^1 \leqq s_c$, in this proposition, implies that the condition, $s_w < s_c$ which is a fundamental assumption of the text of this chapter, is not violated as long as $s_w < s_w^1$. Therefore, the above proposition implies that we are assuming this fundamental assumption holds good as long as there exists the Pasinetti steady state.

However, in Appendices 3, 4, and 5, we will also consider the remaining case where the above assumptions do not hold.

3-1: The Shift of Curve 1

Let us begin the comparative statics by supposing that the economy is initially in the Pasinetti steady state in which the rate of interest is differentiated. This means that we assume $s_c i < n$ and, initially, $i < f'(k^{**})$, that is, $s_w < s_w^1$.

Then, let s_w rise, continuously or discretely, toward the critical level, s_w^1. Let us then see what will happen to the position of the Pasinetti steady state.

For this purpose, we have the following theorem.

THEOREM 2: The position of the Pasinetti steady state moves to the right (left) along Curve 2, when s_w rises (falls, resp.)

Proof. Indeed, we will show that, as s_w rises (falls), Curve 1 shifts to the right, whereas Curve 2 is constant, so that the intersection point between the two curves moves along Curve 2 (Fig. 8).

It is obvious by the form of the function, (20), for Curve 2, that Curve 2 does not shift when s_w changes, since the funcition does not depend on s_w.

Let us then consider the function, $g_1(k)$, defined in (25), for Graph of Curve 1 and show that Curve 1 indeed shifts to the right when s_w rises.

$$g_1(k) = (1/i)[\{nf(k)/(s_c - s_w)\}\{(k/f(k)) - (s_w/n)\} - k\{f'(k) - i\}] \quad (33)$$

Since Curve 1 is upward-sloping, we need to show that this function (33) is a strictly decreasing function of s_w for all k belonging to

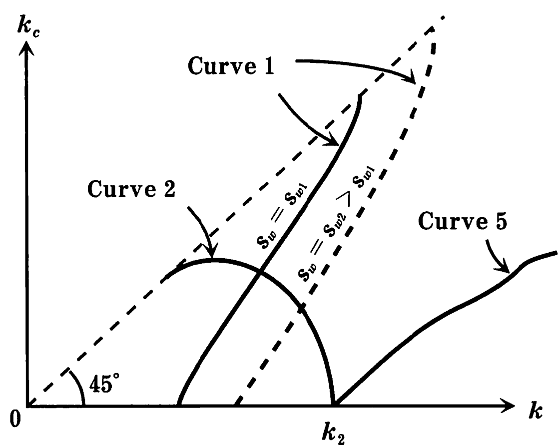

Fig. 8

the domain of Curve 1. For this end, it will suffice for us to verify that

$$\{(k/f(k))-(s_w/n)\}/(s_c-s_w) \qquad (34)$$

is a strictly decreasing function of s_w for all k such that $k_o < k < k_1$. The denominator, s_c-s_w, of this function (34) is always positive by assumption. Let us then show that the numerator of (34) is also positive for all such k.

Since $k^{**}/f(k^{**}) = s_w/n$, we have

$$g_1(k^{**}) < 0 \;\; iff \;\; f'(k^{**}) > i \qquad (35)$$

Therefore, the assumption, $i < f'(k^{**})$, implies $g_1(k^{**}) < 0$. By (29) in Section 2-3, we have $g_1(k_2) > 0$. It follows that the intercept, k_o, of Graph of $\dot{k}$, satisfies $k^{**} < k_o < k_2$.

Hence, for all k belonging to the domain of Curve 1, we have $k^{**} < k$, so that the numerator, $k/f(k)-(s_w/n)$, of (34) is positive over the domain.

Since we have shown that both the denominator and the numerator of (34) are always positive over the domain of Curve 1, it will follow that (34) is a monotonely decreasing function of s_w for all $s_w < s_c$ over the domain, $k_o < k < k_1$ if and only if the function, $\{k/f(k)\}-(s_c/n)$, is negative over the domain.

In order to verify this latter statement, let us prove the following proposition.

PROPOSITION 10: $k_1/f(k_1) < s_c/n$.

Proof. By the definition of k_1 in Section 2-1, we have $g_1(k_1) = k_1$. By (19), it will follow that

$$k_1/f(k_1) = [k_1 f'(k_1)(s_c-s_w)/\{nf(k_1)\}]+(s_w/n) \qquad (36)$$

We have shown in Proposition 5 that the wage-rate is positive for all positive k, so that the relative profit share is less than unity for all $k > 0$. Since $k_1 > 0$, it follows that we have

$$k_1 f'(k_1) < 1 \tag{37}$$

By (36) and (37), we have

$$k_1/f(k_1) < \{(s_c - s_w)/n\} + (s_w/n) = s_c/n \tag{38}$$

Q. E. D.

By this proposition, we know that, if $k \leqq k_1$, then $k/f(k) < s_c/n$. It will follow that (34) is negative over the domain of Curve 1.

Therefore, $g_1(k)$ is a strictly decreasing function of k over the domain of Curve 1. This implies that Curve 1 shifts monotonely downward when s_w increases. By the upward-slopedness of Curve 1, this is equivalent to say that Curve 1 shifts to the right as s_w increases.

Since Curve 1 intersects Curve 2 from below, this implies that the intersection point between Curves 1 and 2 moves to the right along Curve 2. This concludes the proof of Theorem 2. Q. E. D.

3-2: The Shift to the Anti-Pasinetti Steady State

We have clarified how the Pasinetti steady state moves from the initial position, as s_w rises toward s_w^1. Here, it will be in order for us to consider on the state of the economy when $s_w = s_w^1$.

Since we are assuming $s_w^1 \leqq s_c$, there are the two cases, $s_w^1 < s_c$ and $s_w^1 = s_c$.

Suppose, firstly, that $s_w = s_w^1 < s_c$. Then, the strict inequality, $s_w < s_c$, ensures the upward-slopedness of Curve 1, as is the case when the Pasinetti steady state holds.

In this case, we have seen above that $s_w = s_w^1$ is equivalent to say that $k_2 = k^{**}$. Then, by (25), we have $g_1(k_2) = g_1(k^{**}) = 0$, so that, by the definition of k_o, we have $k_2 = k^{**} = k_o$.

$k_2 = k_o$ means that all of the horizontal intercepts, of Graph of $\dot{k}$, Graph of $\dot{k}_c$, and Curve 5, coincide each other, at the point $(k^{**}, 0)$.

DEFINITION 9: The anti-Pasinetti steady state is called *globally stable* if and only if it is converged to over time from any point belonging to the set $\{(k_c, k): 0 \leqq k_c \leqq k < +\infty\} - \{(0, 0)\}$.

PROPOSITION 11: If $s_w = s_w^1 < s_c$, the point, $(k^{**}, 0)$, is the anti-Pasinetti steady state which is unique and globally stable.

Proof. Since the point, $(k^{**}, 0)$, is on both Graph of $\dot{k}$ and Graph of $\dot{k}_c$, it follows that $\dot{k} = \dot{k}_c = 0$. Moreover, at this particular point, we have, of course, $k = k^{**} > 0$ and $k_c = 0$. Hence, this point fulfills all the conditions for the anti-Pasinetti steady state, defined by Definition 2.

In order to strictly verify the uniqueness, we will have to consider the whole phase diagram, taking into consideration not only Curves 1 and 2, but also Curve 5.

Therefore, let us further discuss what properties the phase diagram has, and, by doing so, let us consider both the uniqueness and global stability at the same time.

We have shown in Section 2–4 that Curve 5, starting from the point, $(k^{**}, 0)$, is always upward-sloping. Furthermore, we have shown that Curve 1 is always above Curve 5, if and only if $i \leqq f'(k^{**})$.

Since we are assuming $i = f'(k^{**})$, it follows that Curve 1 is above Curve 5 over the domain of Curve 1.

By analogically utilizing the reasonings of Section 2, concerning the directions of movement of the point initially situated in the separate areas, we can draw the phase diagram in this particular case of $s_w = s_w^1 < s_c$, like that indicated in Fig. 9–a.[3]

By this phase diagram, it will be seen that the anti-Pasinetti steady state at the point, $(k^{**}, 0)$, is indeed unique, and also globally stable. Q. E. D.

It will be shown in Appendix 5 that this proposition can be extended to the case where $s_w^1 = s_c$. The phase diagram in this case looks like that indicated in Fig. 9–b.

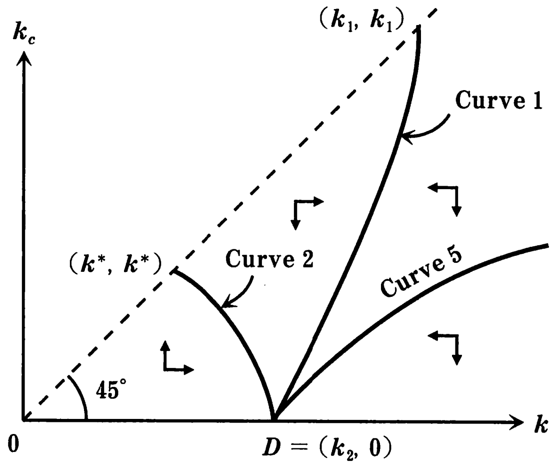

Fig. 9–a

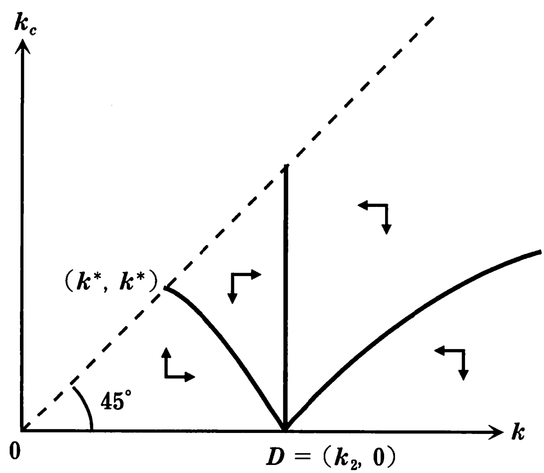

Fig. 9–b

We have proved that, in the process of $s_w^1 s$ increasing from the initial value, say, s_w^o, up to the critical value, s_w^1, the steady state, which corresponds to the value of s_w, keeps to be uniquely determined and globally stable.

This gives us a strong basis on which we can argue what will happen in the process of moving equilibrium in which the steady state shifts along Curve 2, as s_w rises from s_w^o to s_w^1, with both the terminal values, s_w^o and s_w^1, included.

In this process, it is clear by Theorem 2 that the equilibrium overall capital intensity always increases. It follows that the equilibrium overall rate of profit always falls.

Though the equilibrium value of k_c, or the ratio of the capitalists' capital to effective labor, is not necessarily always decreasing, the relative share of the capitalists' share of capital, K_c/K, always falls in the process of moving equilibrium.

Because, Curve 4 is always downward-sloping, which means that the tangent of the straight line from the origin to the point of Curve 2 always falls, as k rises.

For convenience, let us here make the following definitions.

DEFINITIONS 10: For $s_w^o \leqq s_w \leqq s_w^1$, let us define the function, $k_e(s_w)$, as denoting the equilibrium value of k, corresponding to each s_w.

Also, for the same domain, let us define the function, $a(s_w)$, as denoting the capitalists' equilibrium relative share of capital, K_c/K, corresponding to each s_w.

PROPOSITION 12: The functions, $f'(k_e(s_w))$ and $a(s_w)$, are monotonely decreasing. Moreover, we have $\lim_{s_w \to s_w^1 - 0} f'(k_e(s_w)) = i$.

Proof. By $k_e(s_w^1) = k_2$ and $f'(k_2) = i$, we have $f'(k_e(s_w^1)) = i$. Q. E .D.

By this proposition, we can say the reason why the the Pasinetti steady state is transformed into the anti-Pasinetti steady state in this process of moving equilibrium. That is, because the rise in s_w accelerates the societal saving, and increases the overall capital intensity. Thus, it reduces the overall rate of profit down to the differentiated rate of interest, when there does not any longer exist the Pasinetti steady state, but, instead, there does necessarily exist the anti-Pasinetti steady state.

In short, the higher s_w causes the total capital deepening, which brings about the decrease of the overall profit rate down to the rate of interest. Then, the profit rate will be too low for the capitalists to earn their profit at the equilibrium rate which is higher than the rate of interest.

For, since n, s_c, and the production function are constant throughout this process of moving equilibrium, the rate of profit on the capitalists' capital must be equal to n/s_c in equilibrium, which is greater than the rate of interest by assumption. Therefore, by the following relationship, (3), among P/K, P_w/K_w, and P_c/K_c,

$$P/K = (P_w/K_w)(K_w/K)+(P_c/K_c)(K_c/K) \qquad (3)$$

we must have

$$\begin{aligned} f'(k_e(s_w)) &= i(1-a(s_w))+(n/s_c)a(s_w) \\ &= i+\{(n/s_c)-i\}a(s_w) \qquad (39) \end{aligned}$$

all through the process.

Hence, as s_w rises up to s_w^1, the left-hand side decreases down to i, so that, since i and n/s_c are constant, $a(s_w)$ must be reduced to zero.

This is the reason why the capitalists' share of capital, a, must diminish to zero, when s_w rises up to s_w^1.

3-3: Professor Okamoto's Criticism and My Reply

Professor Takeyuki Okamoto of Hannan University criticized the switch mechanism (4), or $P_w = \min\ (iK_w,\ P)$, and claimed that this switch mechanism should be substituted by that of the form $P_w = \min\ (iK_w,\ P-iK_c)$. He argued that it is necessary for P_c/K_c, or the rate of profit of the capitalists' capital, not to be less than i, or the rate of profit of the workers' capital. According to him, the capitalists would not have any motivation to borrow funds from the workers if the capitalists cannot borrow at a cheaper rate of interest than the rate of profit of the capitalists' own funds, that is, if it does not hold that $i \leqq P_c/K_c$, or, equivalently, $iK_c \leqq P_c$. He argued that, if it does not hold that $iK_c \leqq P_c$, or, equivalently, if it does not hold $P - iK_c \geqq P_w$, then the capitalists would not borrow any capital from the workers and the economy would lose much capital for production. The consequent unemployment would cause the economy to enter a drastic recession or even a crisis, according to Professor Okamoto.

So, if his argument were right, it would be necessary that $P_c/K_c \geqq i = P_w/K_w$. This implies that $P/K = (K_w/K)(P_w/K_w) + (K_c/K)(P_c/K_c) \geqq (K_w/K)(P_w/K_w) + (K_c/K)(P_w/K_w) = (P_w/K_w) = i$, that is, $f'(k) \geqq i$.

In reply to Professor Okamato, I would like to point out two things. Firstly, will the capitalists cease to borrow funds from the workers and let the economy enter the crisis so easily? Even if the capitalists' rate of profit falls below the rate of interest at which the capitalists borrow from the workers, will not the capitalists continue to borrow from the workers in order to avoid the fatal event of the crisis which would follow if the capitalists stop borrowing from the workers?

Secondly, even if Professor Okamoto were to be right and even if it were to be necessary that $P_c/K_c < i$, there exists a theoretical possibility that the workers can raise their average propensity to save up to s_w^1 and that the economy can converge to the anti-Pasinetti equilibrium. This is possible by a little-by-little (or quasi-stationary) increase of the workers' propensity to save (s_w) up to s_w^1. The detailed argument is as follows.

In short, it will suffice to show that the workers can raise little by little the average propensity to save up to s_w^1, *without* the point (k, k_c) entering into the area $\{(k, k_c): f'(k) < i\}$, or, equivalently, without the point entering the area $\{(k, k_c): k > k_2\}$, in Figs. 7, 8, and 9–a.

If the average propensity s_w is assumed to be raised not little by little but at one stroke, the dynamic locus of the point (k, k_c) in the adjustment process toward the anti-Pasinetti equilibrium $(k_2, 0)$ can possibly enter the area $\{(k, k_c): k > k_2\}$, as illustrated in Figs. 10–a

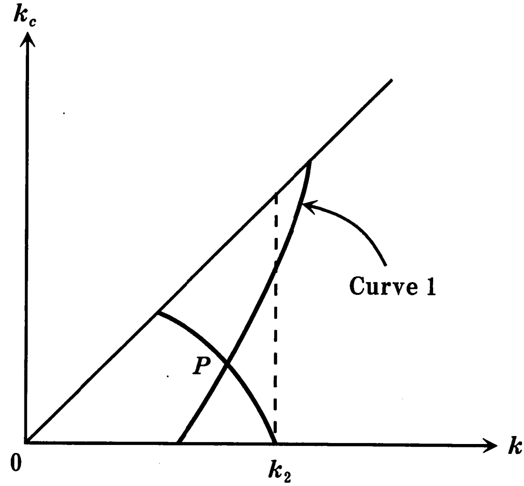

Fig. 10-a

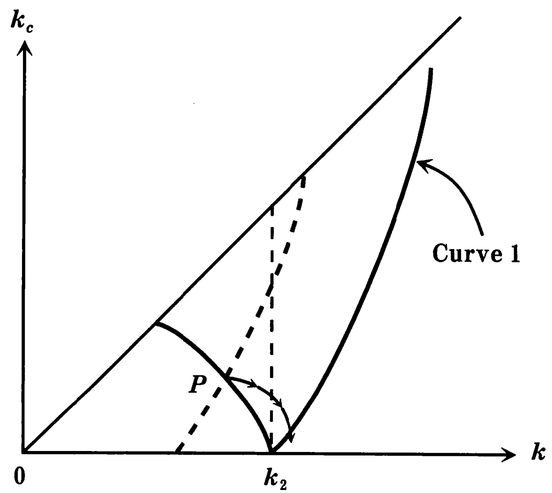

Fig. 10-b

and 10-b. But this possibility of the point entering the area is due to the assumption that s_w is abruptly raised up to s_w^1. If it is assumed that s_w is raised up to s_w^1 little by little, then, as Fig. 10-c shows, the economy can possibly converge to the anti-Pasinetti equilibrium without

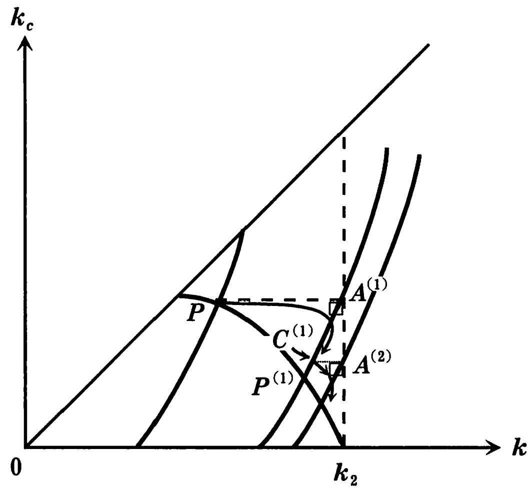

Fig. 10–c

the point entering the area $\{(k,\ k_c):\ k > k_2\}$.

In Fig. 10–c, the point $A^{(1)}$ is the intersection point of the horizontal line which goes through the Pasinetti equilibrium P and the vertical line whose horizontal intercept is k_2. We denote as $s_w^{(1)}$ the level of the workers' average propensity to save at which Curve 1 just goes through the point $A^{(1)}$. The intersection point between the Curve for $s_w^{(1)}$ and Curve 2 is denoted as $P^{(1)}$.

Once s_w is raised up to $s_w^{(1)}$, it is assumed to be constant, until the point $(k,\ k_c)$ is adjusted to be located below the Curve 1 for $s_w^{(1)}$. It is important to see here that the level of k never exceeds k_2 in this adjustment process.

After the point reaches a point, denoted as $C^{(1)}$ in Fig. 10–c, below the Curve 1 for $s_w^{(1)}$, it is assumed that the propensity s_w is again raised from $s_w^{(1)}$ up to $s_w^{(2)}$. The level $s_w^{(2)}$ is defined as the propensity at which Curve 1 goes through the point $A^{(2)}$ which is the intersection point between the horizontal line which goes through $C^{(1)}$ and the vertical line whose horizontal intercept equals k_2. Then, s_w is assumed to be constant at $s_w^{(2)}$, until the point $(k,\ k_c)$ is adjusted to be below the Curve 1 for $s_w^{(2)}$. Remark the point does not enter the area $\{(k,\ k_c):$

$k > k_2\}$ in the adjustment process.

Similarly s_w is assumed to be raised from $s_w^{(2)}$ to $s_w^{(3)}$ which is defined analogously, and from $s_w^{(3)}$ to $s_w^{(4)}$, and so on.

Such a recursive and repeated process of raising s_w and letting the point (k, k_c) be adjusted to cross the shifted Curve 1 from above, the point will converge to the anti-Pasinetti equilibrium $(k_2, 0)$, and in this recursive process, the economy does not enter the crisis area $\{(k, k_c): k > k_2\}$.

APPENDICES

Appendix 3: The Pasinetti Steady States in the Case When $i < \{k_2 f'(k_2)/f(k_2)\}(n/s_c)$

In this appendix, the rate of interest, i, is assumed to be sufficiently small so that i is less than the product of the relative share of capital at $k = k_2$, that is, $k_2 f'(k_2)/f(k_2)$, and the capitalists' equilibrium rate of profit in the Pasinetti steady states, n/s_c.

By Proposition 8, this inequality is equivalent to $s_c < s_w^1$. This latter inequality implies that, as s_w increases toward s_w^1, s_w will reach the level of s_c, and then it will exceed s_c, rising toward s_w^1.

Although, in Chapter 2 of the text of this paper, we have considered the Pasinetti steady states in the case where $s_w < s_c$, we have neglected the case where $s_w \geqq s_c$. Indeed, all through Chapter 2, it was assumed that, even when the workers have become thrifty enough for the anti-Pasinetti steady state to prevail (that is, even when $s_w = s_w^1$), s_w has not attained the level of s_c.

Therefore, when we consider the process of moving equilibrium in which s_w rises from zero toward s_w^1 in this case of $s_w^1 > s_c$ we have to supplement the discussion by considering the Pasinetti and anti-Pasinetti steady states in the case where $s_w^1 \geqq s_w \geqq s_c$.

Let us begin by proving the following theorem concerning the Pasinetti steady state in the case where $s_w \geqq s_c$.

THEOREM a1: Suppose $s_w^1 > s_c$ and $s_w \geqq s_c$. In addition, suppose the production function is of the Cobb-Douglas form, and that the rate of interest is differentiated. Then, the Pasinetti steady state exists, and is uniquely determined, if and only if the rate of interest is less than the dual rate of profit, that is, $i < f'(k^{**})$.

Unfortunately, this theorem seems difficult to verify without the additional assumption of the special, Cobb-Douglas, production function.

Proof. Let us first prove the 'if' part of the above theorem. The outline of the proof of the 'if' part is as follows: under the assumption of $f'(k^{**}) > i$, we draw phase diagrams and, using these diagrams, we analyze the Pasinetti steady state which is represented by the intersection point between Curves 1 and 2. The form of Curve 2 is qualitatively the same as in the case of $s_w < s_c$, since, by (20), the equation for Curve 2 does not depend on s_w. However, the form of Curve 1 will be different, and so the relationship between Curve 1 and Curve 2 has to be examined in order to prove the existence of their intersection point. Let us therefore consider the form and position of Curve 1.

Suppose $s_w > s_c$ (see Appendix 5 for the case of $s_w = s_c$). Then, the function expressing the tangent of the straight line from the origin to each point of Graph of $\dot{k}$ (which includes Curve 1) is the same as (19), which may be rewritten as

$$k_c/k = (1/i)[\{s_w/(s_w - s_c)\}\{f(k)/k - (n/s_w)\} - f'(k) + i] \equiv \phi(k) \qquad \text{(a3)}$$

Let us show that $k^* < k_1 < k^{**}$, where k_1 is the k-coordinate of the intersection point between Graph of $\dot{k}$ and the 45-degree line.

By the definition of k_1, we have $\phi(k_1) = 1$, so that

$$\{s_w/(s_w - s_c)\}\{f(k_1)/k_1\} = f'(k_1) + \{n/(s_w - s_c)\} > n/(s_w - s_c) \qquad \text{(a4)}$$

Hence, $f(k_1)/k_1 > n/s_w = f(k^{**})/k^{**}$. By the decreasing property of the function, $f(k)/k$, this implies $k_1 < k^{**}$.

By the assumption of the Cobb-Douglas production function, we have $f(k_1)/k_1 = Af'(k_1)$, where $A \equiv 1/a > 1$. By (a4), this implies

$$\{s_w/(s_w - s_c)\}Af'(k_1) = f'(k_1) + \{n/(s_w - s_c)\} \qquad \text{(a5)}$$

so that we have

$$f'(k_1) = n/\{(A-1)s_w + s_c\} < n/s_c = f'(k^*) \qquad \text{(a6)}$$

implying $k_1 > k^*$.

This proves $k^* < k_1 < k^{**}$.

k^* is the k-coordinate of the intersection point between Graph of $\dot{k}_c$ (which includes Curve 2) and the 45-degree line. Therefore, $k^* < k_1$ implies that the point, (k_1, k_1), on Graph of $\dot{k}$ lies to the northeast of the point, (k^*, k^*), on Graph of $\dot{k}_c$, both being on the 45-degree line.

The following proposition ensures that the function, $\psi(k)$, of (a3), must be negative for all k, $k^{**} \leqq k \leqq k_2$. It follows that, though, for all k in the neighborhood of k_1, Graph of $\dot{k}$ lies above the horizontal axis, it must lie below the horizontal axis for all k such that $k^{**} \leqq k \leqq k_2$. (By assumption, we have $f'(k^{**}) > i = f'(k_2)$. Hence, $k_2 > k^{**}$.)

PROPOSITION a1: $\psi(k) < 0$ for all k, $k^{**} \leqq k \leqq k_2$ (Fig. A1).

Proof. Let us define $\varphi(k) \equiv \{s_w/(s_w - s_c)\}\{f(k)/k - (n/s_w)\}$ and $\phi(k) \equiv i - f'(k)$. Then, by (a3), we have $\psi(k) = (1/i)\{\varphi(k) + \phi(k)\}$.

Suppose $k^{**} < k < k_2$. then, $k^{**} < k$ implies $n/s_w > f(k)/k$, so that $\varphi(k) < 0$. In addition, $k < k_2$ implies $f'(k) > i$, so that $\phi(k) < 0$. Hence, we have $\psi(k) < 0$.

If $k = k^{**}$, then $\varphi(k) = 0$ whereas, by $k^{**} < k_2$, we have $\phi(k) < 0$. Hence, $\psi(k^{**}) < 0$.

Finally, if $k = k_2$, then, by $k_2 > k^{**}$, we have $\varphi(k) < 0$, whereas we also have $\phi(k_2) = 0$. Hence, $\psi(k_2) < 0$.

This concludes the proof of the proposition.

By this proposition, we know that Graph of $\dot{k}$ not only includes the point, (k_1, k_1), but also lies below the horizontal axis for some k, $k \geqq k^{**}$ (Fig. A1).

On the other hand, Curve 2 is nothing but the part of Graph of $\dot{k}_c$ whose domain is restricted to $k^* < k < k_2$. In addition, we have shown above that $k^* < k_1 < k^{**} < k_2$.

Therefore, by the continuity of the function, $\phi(k)$, for $k_1 \leqq k \leqq k^{**}$, Graph of $\dot{k}$ must intersect Curve 2 at some k, such that $k_1 < k < k^{**}$. Let us denote such an intersection point by P, and its coordinates by (k_e, k_c^e). Then, since P is of course on Curve 2 and since $k_e < k^*$, we have $k_e > k_c^e > 0$. This implies that P is also on Curve 1, because Curve 1 was defined to be the set, $\{(k, k_c): \dot{k} = 0, k > k_c > 0\}$ in Section 1 of Chapter 2.

It follows that the point, P, is an intersection point between Curve 1 and Curve 2. Therefore, P represents a Pasinetti steady state.

This proves, in the case of $s_w > s_c$, that the existence of the Pasinetti steady state is ensured if $i < f'(k^{**})$.

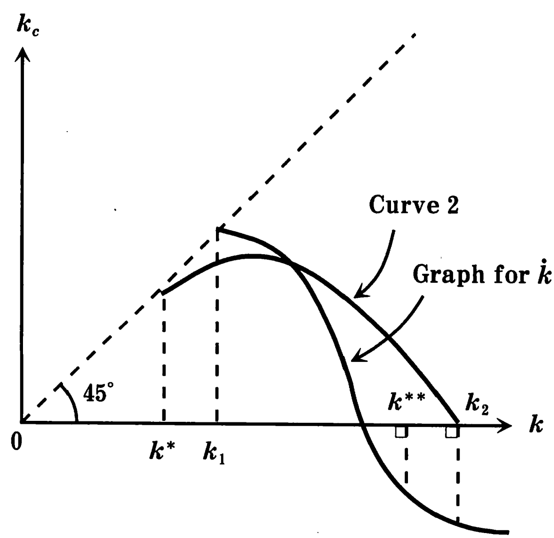

Fig. A1

Let us then verify the uniqueness of the equilibrium.

Since $P = (k_e, k_c^e)$ is on both Curves 1 and 2, it must fulfil both of the equations, (19) (or, equivalently, (a3)) and (21). That is,

$$k_c^e/k_e = (1/i)[\{s_w/(s_w - s_c)\} \{f(k_e)/k_e - (n/s_w)\} - f'(k_e) + i] \quad \text{(a7)}$$

$$k_c^e/k_e = \{s_c/(n - s_c i)\}\{f'(k_e) - i\} \quad \text{(a8)}$$

The equality between the terms on the right-hand sides of (a7) and (a8) can be rearranged as

$$\{s_w/(s_w - s_c)\}\{f(k_e)/k_e - (n/s_w)\} = \{n/(n - s_c i)\}\{f'(k_e) - i\} \quad \text{(a9)}$$

In view of the Cobb-Douglas relationships that $f(k) = K^a L^{1-a}/L = k^a$ and $f(k)/k = k^{a-1}$ and $f'(k) = ak^{a-1}$, (a9) implies

$$[\{s_w/(s_w - s_c)\} - a\{n/(n - s_c i)\}]k_e^{a-1} = -i\{n/(n - s_c i)\} + \{n/s_w - s_c)\} \quad \text{(a10)}$$

Hence, we have

$$k_e^{a-1} = n(n - is_w)/\{s_w(n - is_c) - an(s_w - s_c)\} = n(n - is_w)/\{(1-a)ns_w + s_c(an - is_w)\} \quad \text{(a11)}$$

(a11) shows that, for each given s_w, k_e^{a-1}, and hence, k_e, are uniquely determined, since the term on the right-hand side of (a11) is of course uniquely determined and since k_e^{a-1} is a monotonely decreasing function of k_e.

This proves the uniqueness of the equilibrium of the 'if' part of Theorem 1a, in the case of $s_w > s_c$.

Then, let us proceed to the proof of the 'only if' part of Theorem a1.

Since the Pasinetti steady state is nothing but the intersection point between Curves 1 and 2, we consider the behavior of the intersection point in the process of s_w's continuously increasing from a value less than s_w^1 to a value greater than s_w^1.

In order to facilitate such a consideration, however, it is convenient for us to extend Curves 1 and 2 to Graphs of $\dot{k}$ and $\dot{k}_c$, and to examine the behavior of the intersection point between the graphs.

The k-coordinate of the intersection point between Graphs of $\dot{k}$ and $\dot{k}_c$ must satisfy (a11), since (a11) is derived from (a7) and (a8) which are nothing but the equations expressing Graphs of $\dot{k}$ and $\dot{k}_c$. The solution, k_e, of (a11) represents the k-coordinate of the intersection point between the two graphs. Interpreting k_e in (a11) in this sense, we can prove the following proposition.

PROPOSITION a2: Given the values of all the other parameters than s_w, let us consider the process of moving equilibrium, in which s_w increases. In this process, the intersection point between Graphs of $\dot{k}$ and $\dot{k}_c$ moves to the right, along Graph of $\dot{k}_c$ (Fig. A2).

Proof. Let us regard the right-hand side of (a11) as a function of s_w, and denote it by $u(s_w)$.

Then, the above proposition will be proved if, as we will do in the

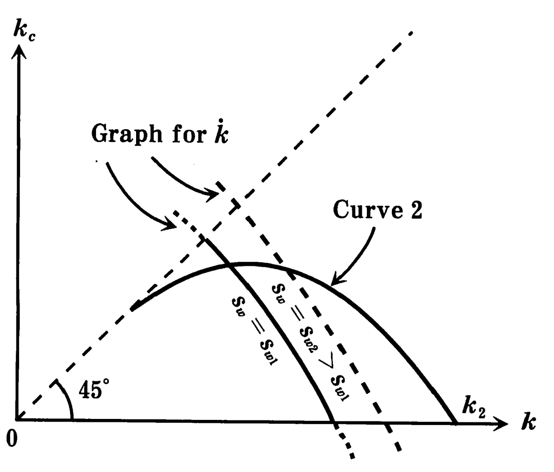

Fig. A2

following, we show that $u(s_w)$ is a monotonely decreasing function, provided(a11) has a solution for k_e.

$u(s_w)$ in (a11) can be rewritten as

$$u(s_w) = n(n-is_w)/[ans_c-\{is_c-(1-a)n\}s_w] \qquad \text{(a12)}$$

Suppose $n-is_w > 0$. Then, both of the numerator and the denominator of the function, $u(s_w)$, are positive. Because, by $-is_w > -n$, the denominator

$$\begin{aligned} ans_c-\{is_c-(1-a)n\}s_w &= ans_c-is_cs_w+(1-a)ns_w \\ &> ans_c-ns_c+(1-a)ns_w \\ &= (1-a)n(s_w-s_c) > 0 \end{aligned} \qquad \text{(a13)}$$

It is very easy to show the monotonely decreasing property of $u(s_w)$ in the case where $is_c-(1-a)n \leqq 0$. Because, if we rewrite

$$u(s_w) = n(n-is_w)/[ans_c+\{(1-a)n-is_c\}s_w] \qquad \text{(a14)}$$

then, the numerator is a decreasing, and the denominator is a (weakly) increasing function, so that $u(s_w)$ is a decreasing function.

In the case where $is_c-(1-a)n > 0$, we first note that $u(s_w)$ is proportional to

$$\{(n/i)-s_w\}/[ans_c/\{is_c-(1-a)n\}-s_w] \qquad \text{(a15)}$$

which can be shown to be a decreasing function of s_w. Because, both of the numerator and the denominator of (a15) are of course positive and, in addition, we have

$$n/i < ans_c/\{is_c-(1-a)n\} \qquad \text{(a16)}$$

Indeed, $(n/i)-[ans_c/\{is_c-(1-a)n\}]$ is proportional to $\{is_c-(1-a)n\}-ais_c = (1-a)(is_c-n)$, which is negative, since $is_c < n$ by assumption.

Remark: In order to verify that (a16) implies the decreasing property of $u(s_w)$, we use the following lemma.

LEMMA a1: Let x be a variable and A and B be constants. Suppose $A-x>0$ for all x. Then, if $A>B$, the function, $(B-x)/(A-x)$, is a decreasing function of x.

Proof of Lemma a1: $(B-x)/(A-x)=1-\{(A-B)/(A-x)\}$. By $A-B>0$ and $A-x>0$, $(A-B)/(A-x)$ is an increasing function of x. Q.E.D.

Suppose $n-is_w \leqq 0$. Then, there does not exist an intersection point between the graphs if the denominator of (a11) is zero or positive. Because, if the denominator of (a11) is zero (positive, resp.), the right-hand side of (a11) is indefinite (negative, resp.), so that (a11) does not have any solution for k_e.

Since we are interested in the movement of the intersection point, we do not have to consider this case in which there is not any intersection point.

Instead, we have to consider the remaining case where the denominator of (18a) is negative.

Since this denominator, of (a11), is of course equal to the denominator of (a12), its negativity implies

$$is_c-(1-a)n>0 \tag{a17}$$

(a12) may be rewritten as

$$n(is_w-n)/[\{is_c-(1-a)n\}s_w-ans_c] \tag{a18}$$

both of the numerator and the denominator of which are positive.

(a18) is proportional to

$$\{s_w-(n/i)\}/[s_w-ans_c/\{is_j-(1-a)n\}] \tag{a19}$$

By (a16) and (a17), it is known that (a18) is a decreasing function of s_w.

Remark: Rigorously, we use the following lemma.

LEMMA a2: Let x be a variable and A and B be constants. Suppose $x-A>0$ for all x. Then, if $A>B$, the function, $(x-B)/(x-A)$, is a decreasing function of x.

Proof of Lemma a2: $(x-B)/(x-A)=1+\{(A-B)/(x-A)\}$. By $A-B>0$ and $x-A>0$, $(A-B)(x-A)$ is a decreasing function of x. Q. E. D.

We have shown that, if the intersection point exists, its k-coordinate always increases as s_w rises.

Since the equation, (20), of Graph of $\dot{k}_c$ does not depend on s_w, Graph of $\dot{k}_c$ is constant. Hence, as s_w rises, the intersection point between Graphs 1 and 2 moves along Graph of $\dot{k}_c$.

This concludes the proof of the proposition.

Let us then consider the form of Graph of $\dot{k}_c$, since we are interested in the position of the intersection point on Graph of $\dot{k}_c$ in the case when $f'(k^{**})\leqq i$, that is when k^{**} is equal to, or greater than, k_2 which is the k-coordinate of the terminal point, or 'end' point, of Curve 2.

It will be shown below that, if $f'(k^{**})>i$, then Curve 2 must be shaped like a hill, rise at first, reach a peak, and then fall (Fig. A3).

Let us verify that Curve 2 always has such a form.

$f'(k^{**})>i$ is equivalent to $af(k^{**})/k^{**}>i$. By $f(k^{**})/k^{**}=n/s_w$, this means $an/s_w>i$, that is, $an-is_w>0$.

By (20), the equation for Curve 2 is

$$k_c=\{s_c/(n-is_c)\}\{f'(k)-i\}k \tag{a20}$$

Let $q(k)\equiv k\{f'(k)-i\}$. Then, we have $q(k)=k\{ak^{a-1}-i)=ak^a-ik$, so that we have $q'(k)=a^2k^{a-1}-i$. Let us define k_4 as the value of k such that $q'(k)=0$.

Then, of course, we have

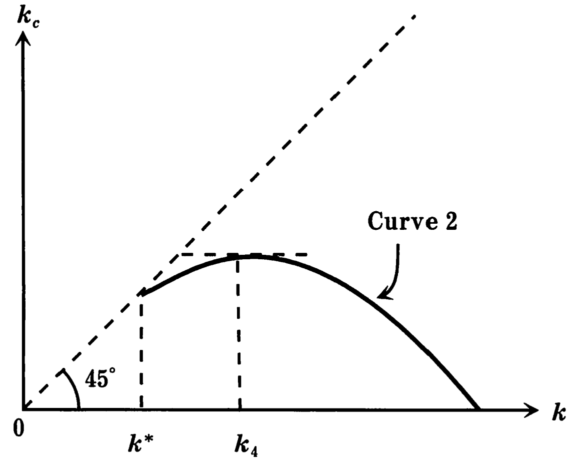

Fig. A3

$$a^2 k_4^{a-1} - i = 0 \tag{a21}$$

In order to consider the form of Curve 2, we are interested in the sign of $q'(k)$ for $k^* < k < k_2$ which is the domain of Curve 2.

Suppose $k = k^*$. Since $ak^{*a-1} = f'(k^*) = n/s_c$, we have $q'(k^*) = (an/s_c) - i = (an - is_c)/s_c$. By $s_w > s_c$, we then have $an - is_c > an - is_w > 0$. Hence, $q'(k^*) > 0$.

Suppose $k = k_2$. Since $ak_2^{a-1} = f'(k_2) = i$, we have $q'(k_2) = ai - i = (a-1)i < 0$.

By the monotonely decreasing property of $q'(k)$, we have $k^* < k_4 < k_2$.

Of course, $q'(k) > 0$ for $k^* < k < k_4$ and $q'(k) < 0$ for $k_4 < k < k_2$, implying that Curve 2 is upward-(downward-, resp.) sloped if $k^* < k < k_4$ ($k_4 < k$, respectively).

Now, let us consider where is the position of the intersection point between the two graphs, when $f'(k^{**}) \leqq i$, i. e., $k^{**} \geqq k_2$.

Suppose $k^{**} = k_2$. In this case, Graph of $\dot{k}$ intersects the horizontal axis at $k = k_2$, because, by regarding (a7) as the equation for Graph of $\dot{k}$, the k_c-coordinate of Graph of $\dot{k}$ for $k = k_2$ equals zero. Indeed,

$k^{**} = k_2$ means $n/s_w = f(k_2)/k_2$, so that, by this equality and $f'(k_2) = i$, we know that the right-hand side of (a7) vanishes if k_e of (a7) is substituted by k_2.

It follows that both of the two graphs go through the point, $(k_2, 0)$. This means that this particular point, $(k_2, 0)$ is the intersection point between these graphs in the case when $k^{**} = k_2$.

Suppose $k^{**} > k_2$. This implies $s_w/n = k^{**}/f(k^{**}) > k_2/f(k_2)$. This means that s_w in the case of $k^{**} > k_2$ is greater than s_w in the case of $k^{**} = k_2$. Because the last term, $k_2/f(k_2)$, equals s_w in the case of $k^* = k_2$, divided by n.

Since the intersection point between the graphs moves to the right as s_w rises, it follows that the intersection point in the case of $k^{**} > k_2$ lies to the right of that in the case of $k^{**} = k_2$.

Therefore, we can conclude that, if $f'(k^{**}) \leqq i$, there does not exist any intersection point between Curve 1 and Curve 2.

Indeed, if $f'(k^{**}) = i$ we have, of course, $k^{**} = k_2$, and this implies that Graph of $\dot{k}$ (as distinct from Curve 1) and Graph of $\dot{k}_c$ (as distinct from Curve 2) intersect each other at a point (which is $(k_2, 0)$) on the horizontal axis, at which Curves 1 and 2 are not defined.

Furthermore, if $f'(k^{**}) < i$, we have $k^{**} > k_2$. As seen above, this implies that the intersection point between the graphs is located to the right of that in the case of $k^{**} = k_2$, which, we now know, is $(k_2, 0)$. Since Graph of $\dot{k}_c$ is below the horizontal axis for $k > k_2$, the intersection point between the graphs is below the horizontal axis, and hence it cannot be an intersection point between Curves 1 and 2. It follows that Curves 1 and 2 do not intersect in this case of $f'(k^{**}) < i$.

This proves, in the case of $s_w > s_c$, the 'only if' part of Theorem a1.

Appendix 4: The Stability in the Case When $s_w^1 > s_w > s_c$

Here, we will consider the stability of the Pasinetti steady state in the case when $s_w^1 > s_w > s_c$.

Unfortunately, in this case, the global stability of the Pasinetti steady state does not seem to be always ensured. However, its local stability is always ensured in this case (Section 3 of this appendix).

However, we will present a sufficient condition for its global stability in this case.

Let us summarize some results obtained in the above appendix concerning the existence and uniqueness of the Pasinetti steady state in this case.

We have clarified the following points: Under the assumptions of the Cobb-Douglas production function, $s_w > s_c$, $n > is_c$, and $i < f'(k^{**})$, (1) the Pasinetti steady state necessarily exists and is uniquely determined, and (2) Curve 1 intersects Curve 2 from the area above Curve 2.

Let us further explain the second point.

We have shown above that $k^* < k_1$, that is, Curve 1 intersects the 45-degree line always at a point (which is (k_1, k_1)) to the north east of the intersection point between Curve 2 and the 45-degree line (which is (k^*, k^*)). We have also shown that Curve 1 is below the horizontal axis for $k^{**} \leqq k \leqq k_2$.

These facts together imply the above second point, since it has been proved there that $k^* < k^{**}$ even in this case of $s_w > s_c$.

However, it should be remarked that the slope of Curve 1 at the intersection point with Curve 2 may be positive, zero, or negative, so that the global stability cannot be generally ensured.

Indeed, as considered just below, if Curve 1's slope is negative at the point of intersection, the equilibrium is necessarily globally stable, but if not, it may not be globally stable.

1. The Global Stability

Suppose Curve 1's slope at the intersection point is negative. Then, we have phase diagrams like those indicated in Figs. A4–a and A4–b. (By equations (7) and (8) in the text, it is easy to check that the point moves to the left (right) when it is located above (below, respectively) Curve 1, in this case of $s_w > s_c$.) Figs. A4–a, b show the

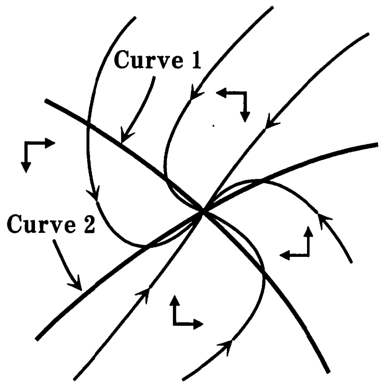

Fig. A4-a

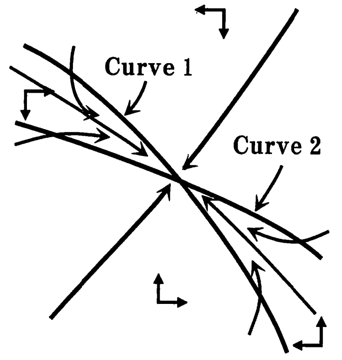

Fig. A4-b

following proposition.

PROPOSITION a3: The Pasinetti equilibrium is globally stable if the slope of Curve 1 is negative at the equilibrium, irrespective of the sign of the Curve 2's slope there.

However, as indicated in Figs. A5-a, b and c, the equilibrium may well be unstable if Curve 1's slope is zero or positive at the equilibrium.

Both Fig. A5-a and b depict the case when Curve 1 is positively sloped at the intersection, the former is the case of divergence and the latter the case of a limit cycle.

Fig. A5-c indicates the case in which the equilibrium is globally stable, although the slope of Curve 1 is not negative.

Though Curve 1 is positive-sloped in all of these Figs. A5-a, b, and c, it is easy to check that the qualitatively similar results also apply to the case when its slope is zero at the equilibrium.

Incidentally, Figs. A6-a and b depict examples of the impossible cases in which Curve 1 crosses Curve 2 from the area below Curve 2, which are excluded.

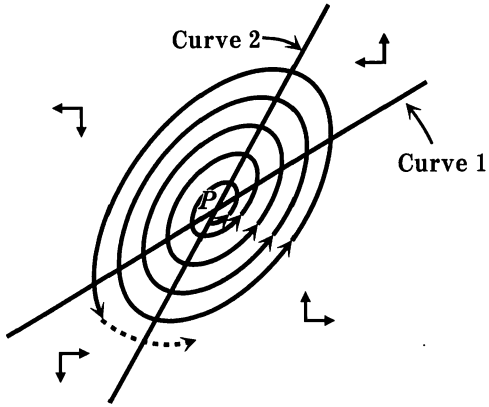

Fig. A5-a

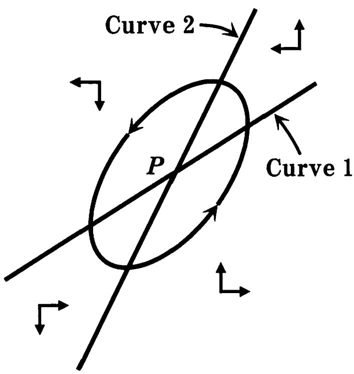

Fig. A5-b

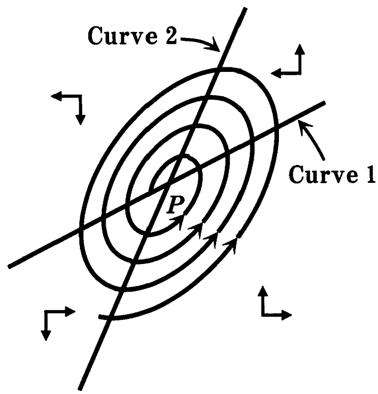

Fig. A5-c

2. A Sufficient Condition for the Global Stability

In this section, we will prove the following theorem.

THEOREM a2: The Pasinetti steady state is globally stable if $i > a^2(n/s_c)$.

Proof. As shown above, Curve 2 is shaped like a hill, and the sign of the slope of Curve 2 is the same as that of the function $q'(k) =$

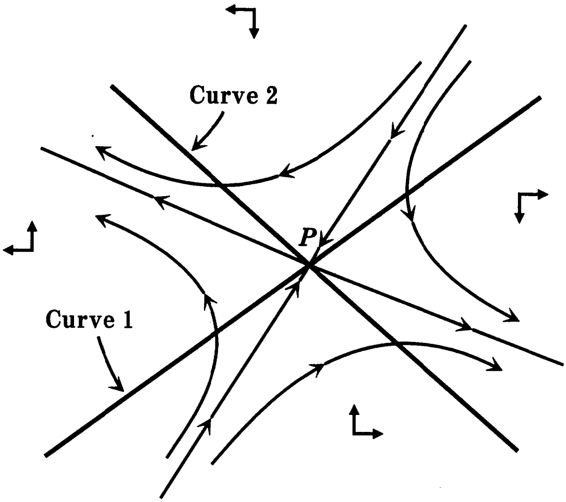

Fig. A6–a

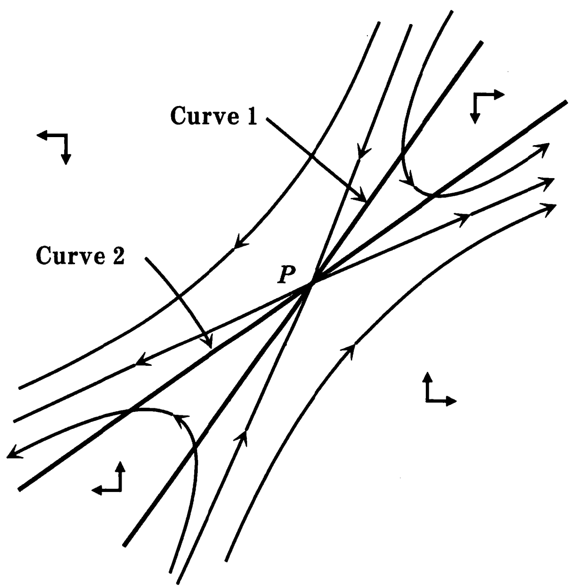

Fig. A6–b

$a^2k^{a-1}-i$. (See (a20).) In addition, we have verified that Curve 1 intersects Curve 2 always from the area above Curve 2.

It follows that, if the intersection point between the two curves is to the right of the peak of Curve 2, the global stability obtains.

Because the intersection point is in such a position if and only if Curve 2 is negatively-sloped at the equilibrium. Furthermore, the fact that Curve 1 crosses Curve 2 always from the area above the latter does imply that the slope of Curve 1 is *algebraically less* than Curve 2 at the equilibrium (Fig. A4-b).

Therefore, if they cross each other to the right of the peak of Curve 2, then it follows that the slope of Curve 1 must be also negative. By Proposition a3 in this appendix, this implies the global stability of the Pasinetti equilibrium.

Let us then prove the following lemma a3.

LEMMA a3: The equilibrium is to the right of the peak if $i > a^2(n/s_c)$.

It is easy to see that if we can verify this lemma, then the main proposition of this section will be completely proved.

Proof of Lemma a3: Consider the vector of the parameters of our model, (a, n, i, s_c). We have shown that, with this vector given, the equilibrium moves to the right along Curve 2 as s_w increases. (See Proposition a2.)

It follows that, for the fixed vector, if s_w is just equal to the fixed value of s_c when the equilibrium is located to the right of the peak of Curve 2, then the equilibrium for all $s_w > s_c$ is always to the right of the peak.

We will show that the condition, $i > a^2(n/s_c)$, is necessary and sufficient for the equilibrium for $s_w = s_c$ to be to the right of the peak.

If $s_w = s_c$, Curve 1 is reduced to a vertical line at $k = k^{**}$, because, as will be shown in Appendix 5 below, we have $\dot{k} = s_w f(k) - nk$, so

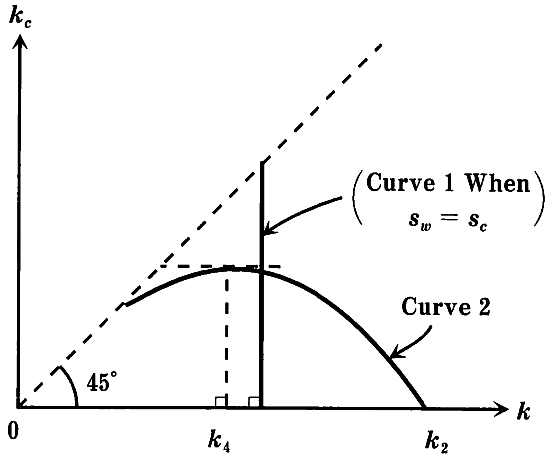

Fig. A7

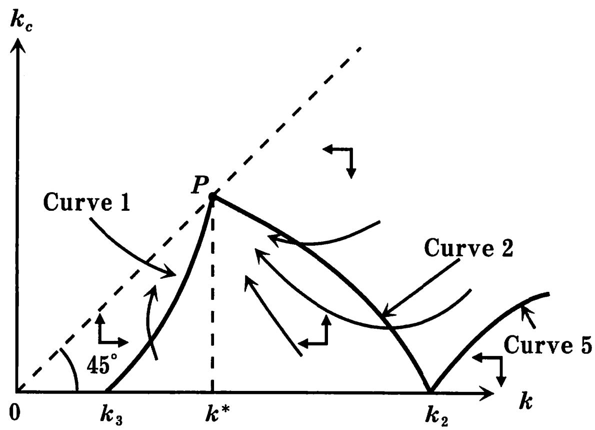

Fig. A8

that $\dot{k} = 0$ implies $s_w/n = k/f(k)$, or $k = k^{**}$ (Fig. A7).

The k-coordinate of the equilibrium for $s_w = s_c$ is therefore, of course, k^{**}. So, the sign of the slope of Curve 2 at the equilibrium for $s_w = s_c$ is given by $q'(k^{**}) = a^2 k^{**a-1} - i$.

By $f(k^{**})/k^{**} = n/s_w$, we have $n/s_w = f(k^{**})/k^{**} = (k^{**})^a/k^{**} = (k^{**})^{a-1}$. It follows that $q'(k^{**}) = a(n/s_w) - i = a^2(n/s_c) - i$.

This proves that $i > a^2(n/s_c)$ if and only if the equilibrium for $s_w = s_c$ is located to the right of the peak of Curve 2. Lemma a3, Q.E.D.

Hence, $i > a^2(n/s_c)$ is a sufficient condition for the global stability of the Pasinetti equilibrium for $s_w > s_c$.

We should remark here that the only partly analogous inequality, $i < a(n/s_c)$, has been implicitly assumed in this appendix. Indeed, this inequality means nothing but $s_c < s_w^1$. (See the proof of Theorem 4.)

Since $1 > a$, we have $a(n/s_c) > a^2(n/s_c)$, so that the above result allows us to say that the global stability of the Pasinetti equilibrium for $s_w > s_c$ is ensured if $a(n/s_c) > i > a^2(n/s_c)$.

3. The Local Stability

In the last section, we have shown that the global stability is ensured if the slope of Curve 2 at the equilibrium is negative. We have also presented a sufficient condition for the slope of Curve 2 at the equilibrium to be indeed negative.

Then, what can be said about the stability if the slope of Curve 2 in equilibrium is zero or positive? Is there any sufficient condition for the global stability even in the case in which the slope is not negative?

In this case when $q'(k_e) \geqq 0$ ($q'(k_e)$ denotes the slope $q'(k)$ of Curve 2 at the equilibrium value k_e of k), such a condition has not been obtained. However, as will be shown below, the local stability is always ensured.

By the assumption of the Cobb-Douglas production function, equations (7), and (8) in the text may be rewritten as

$$\begin{aligned} \dot{k} &= (s_w - s_c)\{ik - kf'(k) - ik_c\} + s_w f(k) - nk \\ &= (s_w - s_c)\{ik - k(ak^{a-1}) - ik_c\} + s_w k^a - nk \\ &= \{s_w - (s_w - s_c)a\}k^a + \{(s_w - s_c)i - n\}k - (s_w - s_c)ik_c \quad \text{(a22)} \end{aligned}$$

$$\begin{aligned}\dot{k}_c &= s_c\{(f'(k)-i)k+ik_c\}-nk_c \\ &= s_c\{(ak^{a-1}-i)k+ik_c\}-nk_c \\ &= as_ck^a-is_ck+(is_c-n)k_c \end{aligned} \tag{a23}$$

Since (k_e, k_c^e) denote the equilibrium point, $\dot{k} = \dot{k}_c' = 0$ at (k_e, k_c^e). Therefore, we can use the classical method of making the system of linear differential equations with respect to the variables, (h, h_c) (where $h = k-k_e$ and $h_c = k_c-k_c^e$) which locally approximates the above non-linear system of differential equations with respect to (k, k_c) around the equilibrium point, (k_e, k_c^e).

Thus, we have

$$\begin{pmatrix}\dot{h}\\ \dot{h}_c\end{pmatrix} = \begin{pmatrix}a_{11}, & a_{12}\\ a_{21}, & a_{22}\end{pmatrix}\begin{pmatrix}h\\ h_c\end{pmatrix} \tag{a24}$$

where

$$a_{11} = a\{s_w-(s_w-s_c)a\}k_e^{a-1}+\{(s_w-s_c)i-n\}, \quad a_{12} = -(s_w-s_c)i$$
$$a_{21} = a^2s_ck_e^{a-1}-is_c, \quad a_{22} = is_c-n$$

The characteristic polynomial of the matrix $[a_{ij}]$ can be written in the form, $B(x) = x^2+B_1x+B_0$, where

$$B_1 = -[a\{s_w-(s_w-s_c)a\}k_e^{a-1}+\{(s_w-s_c)i-n\}+is_c-n] \tag{a25}$$
$$\begin{aligned}B_0 = {} & [a\{s_w-(s_w-s_c)a\}k_e^{a-1}+\{(s_w-s_c)i-n\}](is_c-n) \\ & +(s_w-s_c)i(a^2s_ck_e^{a-1}-is_c)\end{aligned} \tag{a26}$$

We will show that B_0 is always positive.

B_0 can be arranged into the form, $B_0 = C_1k_e^{a-1}+C_0$, where C_0 does not include any powers of k_e.

$$C_1 = a\{s_w-(s_w-s_c)a\}(is_c-n)+(s_w-s_c)ia^2s_c, \tag{a27}$$
$$C_0 = \{(s_w-s_c)i-n\}(is_c-n)+(s_w-s_c)i(-is_c) \tag{a28}$$

It is easy to see that

$$C_0 = -(s_w-s_c)in-n(is_c-n) = -s_win+n^2 = n(n-is_w). \tag{a29}$$

Also, we have

$$\begin{aligned} C_1 &= a[s_w(is_c-n)-(s_w-s_c)a(is_c-n)+(s_w-s_c)ais_c] \\ &= a\{s_w(is_c-n)+(s_w-s_c)an\} \\ &= a\{(a-1)ns_w+s_c(is_w-an)\} \\ &= -a\{(1-a)ns_w+s_c(an-is_w)\} \end{aligned} \tag{a30}$$

By (a11), we have

$$k_e^{a-1} = n(n-is_w)/\{(1-a)ns_w+s_c(an-is_w)\}, \tag{a31}$$

so that we have

$$C_1 k_e^{a-1} = -an(n-is_w). \tag{a32}$$

It follows that

$$B_0 = -an(n-is_w)+n(n-is_w) = (1-a)n(n-is_w). \tag{a33}$$

By assumption, we have $i < f'(k^{**})$. Since k^{**} has been defined by $k^{**}/f(k^{**}) = s_w/n$, we also have $f'(k^{**}) = a(k^{**})^{a-1} = a(k^{**})^a/k^{**} = af(k^{**})/k^{**} = a(n/s_w)$. Hence, we have $i < an/s_w$, so that $i < n/s_w$ by $a < 1$. Hence $n-is_w > 0$. It follows that B_0 is always positive.

The equilibrium point (0, 0) of the system of linear differential equations with respect to (h, h_e) is stable if and only if $B_0 > 0$ and $B_1 > 0$. Since we have shown that $B_0 > 0$, the equilibrium is stable if and only if $B_1 > 0$.

Since the equilibrium of the original system of differential equations for (k, k_c) is called locally stable if and only if that of the linear (approximating) system of differential equations for (h, h_c) is stable, it follows that $B_1 > 0$ is a necessary and sufficient condition for the original system to be locally stable.

PROPOSITION a4: As long as $i < f'(k^{**})$, the Pasinetti steady state for $s_w > s_c$ is always locally stable.

Proof. The tangent of the straight line from the origin to the point on Curve 2 is a decreasing function of k. In addition, Curve 2 is always below the 45-degree line. It follows that the slope of Curve 2 at the equilibrium is less than unity.

Therefore, the slope of Curve 1 at the equilibrium is also less than unity since the slope of Curve 1 is algebraically less than that of Curve 2. (See Section 2 of this appendix.)

By (18), the equation for Curve 1 is

$$\begin{aligned} k_c &= (1/i)[\{nf(k)/(s_w-s_c)\}\{(s_w/n)-(k/f(k))\}-k\{f'(k)-i\}] \\ &= (1/i)[\{nk^a/(s_w-s_c)\}\{(s_w/n)-k^{1-a}\}-k\{ak^{a-1}-i\}] \\ &= (1/i)[\{s_w/(s_w-s_c)\}k^a-\{n/(s_w-s_c)\}k-ak^a+ik] \\ &= (1/i)[\{s_w/(s_w-s_c)\}-a]k^a+(1/i)[i-\{n/(s_w-s_c)\}]k \\ &= [\{s_w-a(s_w-s_c)\}k^a+\{i(s_w-s_c)-n\}k]/\{i(s_w-s_c)\} \end{aligned} \quad \text{(a34)}$$

Hence, the slope of Curve 1 is expressed as

$$[a\{s_w-a(s_w-s_c)\}k^{a-1}+\{i(s_w-s_c)-n\}]/\{i(s_w-s_c)\} \quad \text{(a35)}$$

Since the slope of Curve 1 at k_e is less than unity, we have

$$[a\{s_w-a(s_w-s_c)\}k_e^{a-1}+\{i(s_w-s_c)-n\}]/\{i(s_w-s_c)\} < 1, \quad \text{(a36)}$$

so that we have

$$-[a\{s_w-a(s_w-s_c)\}k_e^{a-1}+\{i(s_w-s_c)-n\}] > -i(s_w-s_c) \quad \text{(a37)}$$

That is,

$$-a\{s_w-a(s_w-s_c)\}k_e^{a-1}+n > 0. \quad \text{(a38)}$$

It follows that

$$-[a\{s_w-a(s_w-s_c)\}k_e^{a-1}-is_w+2n > n-is_w \quad \text{(a39)}$$

The left-hand side as a whole is equal to B_1. Because

$$\begin{aligned} B_1 &= -[a\{s_w-(s_w-s_c)a\}k_e^{a-1}+\{(s_w-s_c)i-n\}+is_c-n] \\ &= -a\{s_w-a(s_w-s_c)\}k_e^{a-1}-is_w+2n \end{aligned} \quad \text{(a40)}$$

Thus, we have $B_1 > n-is_w$.

Now, we have seen that $i < f'(k^{**})$ implies $n - is_w > 0$. Hence, $B_1 > 0$. Q. E. D.

Appendix 5

We consider here the special case in which $s_w = s_w^1 = s_c$.

In this case, equation (7) is reduced to the form, $\dot{k} = s_w f(k) - nk$. Thus, Curve 1 becomes a vertical line at $k = k^{**}$, whereas the form of Curve 2 is the same as in the case of $s_w^1 < s_c$. The phase diagram in this case looks like that indicated in Fig. 9-b, and the equilibrium point, $(k_2, 0)$, is a globally stable, anti-Pasinetti steady state, as in the case of $s_w^1 < s_c$.

FOOTNOTES

3 Like what will look the phase diagram in the case when s_w further rises from the level of s_w^1 and *exceeds* s_w^1, but still is less than s_c?

In this note, we will briefly consider this case of $s_c > s_w > s_w^1$.

We have seen in p. 32 of this paper No. I in this journal, the *Keizai Shirin*, Vol. 64, No. 3 that Curve 1 and Curve 5 intersect at $k = k^{**}$. Also, in view of

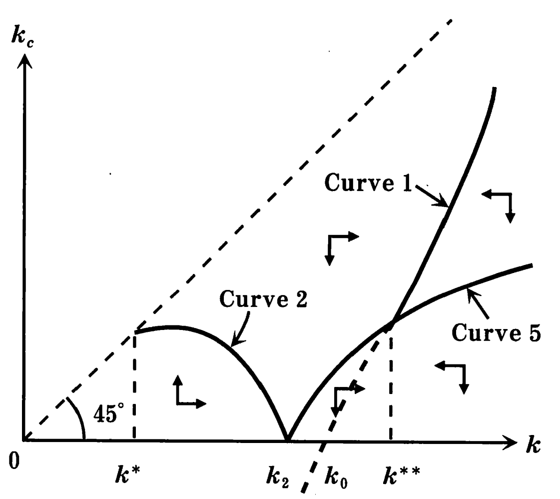

Fig. N1

(32) and the argument in p. 34 of the *Keizai Shirin,* Vol. 64, No. 3, it can be easily verified that, in the area to the right of Curve 5 (i. e., such that $iK_w > P$), we have $\dot{k} > 0$ if $k < k^{**}$ and $\dot{k} < 0$ if $k > k^{**}$. Hence, the phase diagram in this case will look like that in Fig. N1.

ERRATUM

The following sentence should be deleted. "It follows that ... and 6–b ..." in Line 11–9 from below, p. 36, Part I of this article in the *Keizai Shirin,* Vol. 64, No. 3.